Comprehensive Exam Proposal

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Date: October 25, 2016 2:30pm

Factorization in Commutative Rings with Zero Divisors

Abstract

This talk will discuss how the theory of factorization in integral domains generalizes to commutative rings with zero divisors. Let R be a commutative ring. Two elements $a, b \in R$ are said to be associates, denoted $a \sim b$, if $(a) = (b)$. Then $a \in R$ is irreducible if $a = bc$ implies $a \sim b$ or $a \sim c$. This is the weakest of several types of irreducibility, several of which depend on different types of associate conditions. R is said to be atomic if every nonzero nonunit can be written as the finite product of irreducibles. Thus each type of irreducible leads to a different form of atomicity. We will prove a maximality condition for a zero divisor to be irreducible. We will also discuss whether certain properties extend to polynomial rings.

Proposal

The focus of my comprehensive exam will be factorization in commutative rings with zero divisors. To gain background in commutative ring theory I read and completed select exercises from the first three chapters of *Commutative Ring Theory* by Irving Kaplansky [5]. Topics in this text include prime ideals, integral domains, localization, Noetherian rings, quasi-local domains, R-sequences, and regular rings. To get an overview of how factorization works in integral domains I read the paper *Factorization in Integral Domains* by D.D. Anderson, D.F. Anderson, and Muhummad Zafrullah [1]. To see how this theory generalizes to commutative rings I read *Factorization in Commutative Rings with Zero Divisors* by D.D. Anderson and S. Valdes-Leon [2]. I also read two additional papers *Irreducible Elements in Commutative Rings with Zero Divisors* [3] and *Irreducible Elements in Commutative Rings with Zero Divisors, II* [4] by D.D. Anderson and S. Chun. These expanded upon and made concise the results of the paper on general factorization in commutative rings with zero divisors with a focus on irreducible elements and atomicity.

I will begin the talk with a series of definitions. In an integral domain we say that $a \in R$ is *irreducible* if $a = bc$ implies that $b \in U(R)$ or $c \in U(R)$ where $U(R)$ is the group of units. We say that a and b are *associated*, denoted $a \sim b$, if $a \mid b$ and $b \mid a$, or equivalent if $(a) = (b)$. Then we have that the following are equivalent: (1) a is irreducible (2) $a = bc \implies a \sim b$ or $a \sim c$ (3) (a) is maximal in the set of proper principal ideals of R. In a commutative ring R it is a question how to define an irreducible element. Several authors have come up with different definitions. The goal of [2] was to provide a unifying theory on factorization in commutative rings with zero divisors. To begin our discussion of irreducible elements I will define several associate conditions. For $a, b \in R$ we say that a and b are *strongly associated*, denoted $a \approx b$, if $a = ub$ for some $u \in U(R)$. We say that a and b are *very strongly associated*, denoted $a \cong b$, if (1) $a \sim b$ and (2) $a = b = 0$ or $a \neq 0$ and $a = rb \implies r \in U(R)$. Each associate condition leads to a different type of irreducible element. Thus $a \in R$ is *irreducible* (resp. *strongly irreducible*, resp. *very strongly irreducible*) if $a = bc$ implies that $a \sim b$ or $a \sim c$ {resp. $a \approx b$ or $a \approx c$, resp. $a \cong b$ or $a \cong c$}. We also say that $a \in R$ is *m-irreducible* if (a) is maximal in the set of proper principal ideals. As in the domain case, $p \in R$ is prime if $p \mid ab \implies p \mid a$ or $p \mid b$.

In my talk I will show that the above leads to the following implications and provide counterexamples to show that none of these implications can be reversed.

$$\begin{tikzcd} & & & \text{prime} \arrow[d, Rightarrow] \\ \text{very strongly irreducible} \arrow[r, Rightarrow] & m\text{-irreducible} \arrow[r, Rightarrow] & \text{strongly irreducible} \arrow[r, Rightarrow] & \text{irreducible} \end{tikzcd}$$

Note that unlike the domain case, irreducible $\not\Longrightarrow$ m-irreducible. However there is a maximality condition for commutative rings. An element $a \in R$ is irreducible if and only if (a) is maximal in the set of proper principal ideals contained within some fixed prime P. This says that if P is a prime ideal of R consisting of zero-divisors and (a) is maximal in the set of proper principal ideals contained in P, then a is a zero-divisor that is irreducible. I will prove that the converse is also true, that is if a is a zero-divisor that is irreducible, then there is a prime ideal P consisting of zero divisors where (a) is maximal in the set $\{(b) \mid b \in P\}$. Next we will have a brief discussion about atomicity. A commutative ring R is said to *atomic* (resp. *strongly atomic*, resp. *very strongly atomic*, resp. *m-atomic*, resp.*p-atomic*) if any nonzero nonunit in R can be written as the finite product or irreducible (resp. strongly irreducible, resp. very strongly irreducible, resp. m-irreducible, resp. prime) elements. Like in the domain case, ACCP implies atomic.

For the time that remains in my talk I will discuss polynomial rings. I will discuss the behavior of associate relations and list some results about irreducibles. For example, a in R is irreducible iff a is irreducible in $R[\{X_\alpha\}]$. This result does not hold for stronger versions of irreducible. Lastly, it is of some interest which properties defined on R can be extended to $R[X]$ or $R[[X]]$. Many do not. For instance R atomic does not imply $R[X]$ is atomic. R is a bounded factorization ring (BFR) if for each nonzero nonunit $a \in R$ there exists a natural number $N(a)$ so that for any factorization $a = a_1 \dots a_n$ where each a_i is a nonunit we have $n \leq N(a)$. In general R a BFR does not imply that $R[X]$ is a BFR. If time permits, some results on finite factorization rings (FFRs) and weak finite factorization rings (WFFRs) will be discussed.

Ideas going forward include investigating what has been done on polynomial rings with zero divisors and developing a unifying theory that expands upon these results, similar to what was done in [1] and [2]. It is an open question whether R satisfies ACCP implies $R[[X]]$ satisfies ACCP, though W. Heinzer and D. Lantz have shown R ACCP does not imply $R[X]$ satisfies ACCP.

References

[1] D.D. Anderson, D. F. Anderson, and Muhammad Zafrullah *Factorization in integral domains* Journal of Pure and Applied Algebra 69 (1990), 1-19.

[2] D. D. Anderson and S. Valdes-Leon *Factorization in commutative rings with zero divisors,* Rocky Mountain J. Math 26 (1996), 439-480.

[3] D. D. Anderson and S. Chun *Irreducible elements in commutative rings with zero divisors,* Houston J. Math. 37 (2011), 741-744.

[4] D. D. Anderson and S. Chun *Irreducible elements in commutative rings with zero-divisors, II,* Houston J. Math. 39 (2013), 741-752.

[5] Irving Kaplansky *Commutative Rings* The University of Chicago Press (1974)