§ 1-5 PRIME IDEALS IN POLYNOMIAL RINGS

Theorem 36: Let R be an integral domain with quotient field K, and let X be an indeterminate. Then we have the following correspondence:

$$\left\{ \begin{matrix} \text{prime ideals in } R[X] \\ \text{that contract to } 0 \text{ in } R \end{matrix} \right\} \Longleftrightarrow \left\{ \text{prime ideals in } K[X] \right\}$$

proof: Let S be the set of nonzero elements in R. Then,

$$R_S = \left\{ \frac{r}{s} \;\middle|\; s \in R \text{ with } s \neq 0 \right\} = K$$

where K is the quotient field of R. Also note that,

$$R[X]_S = \left\{ \frac{f}{g} \;\middle|\; g \in R[X] \text{ with } g \neq 0 \right\} = K[X].$$

We know there is a one-to-one correspondence between prime ideals in $K[X]$ and primes in $R[X]$ disjoint from S, that is, prime ideals in $R[X]$ that intersect 0, that is prime ideals that contract to 0 in R. $\square$

Remark: We can say some things about prime ideals in $K[X]$ and $R[X]$.

primes in $K[X]$

↳ infinitely many maximal ideals corresponding to irreducible polynomials over K

primes in $R[X]$

↳ for $P \subset R$ prime, we can expand to $PR[X] \subset R[X]$ prime

↳ $PR[X]$ and infinitely many prime ideals sitting above $PR[X]$ contract to P

Note: By Theorem 37 there cannot be a chain of three distinct prime ideals with the same contraction in R.

def: Let $P = P_0 \supset P_1 \supset \cdots \supset P_n$ be a chain of distinct prime ideals descending from P. This is a chain of length n. We say P has rank n if there exists a chain of length n descending from P, but no longer chain. If there exist arbitrarily long chains descending from P we say P has rank ∞.

Remark: the rank of a prime P is also called the height or altitude

Note: A minimal prime has rank 0. In an integral domain since 0 is prime rank 1 primes are called minimal.

Theorem 38: Let P be a prime ideal of rank n in R. In the polynomial ring $R[X]$, write $P^* = PR[X]$, and let Q be a prime ideal that contracts to P in R and contains P^* properly. Then,

(a) $n \leq \text{rank}(P^*) \leq 2n$, and

(b) $n+1 \leq \text{rank}(Q) \leq 2n+1$.

proof: Let $P = P_0 \supset P_1 \supset \cdots \supset P_n$ be a chain of primes descending from P. Consider the expansion of each P_i to $R[X]$, i.e. $P_i^* = P_i R[X]$. Then we get the chain $Q \supset P_0^* \supset P_1^* \supset \cdots P_n^*$. Then $\text{rank}(P^*) \geq n$ and $\text{rank}(Q) \geq n+1$. Now consider a chain descending from $P^* = P_0^* \supset P_1' \supset \cdots P_k'$. There cannot exist a chain of three distinct prime ideals in $R[X]$ contracting to the same prime ideal P' in R. Only P^* contracts to P, and the others contract at most two to one to primes in R. Thus, $\text{rank}(P^*) \leq 2n$ and similarly the $\text{rank}(Q) \leq 2n+1$. $\square$

EXERCISES

① Let Q be prime ideal in $R[X]$, contracting to P in R. Prove that Q is a G-ideal *iff* P is a G-ideal and Q properly contains $PR[X]$.

tools

def: Let R be an integral domain with quotient field K. Then R is a G-domain if TFAE:

(1) K is a finitely generated ring over R

(2) as a ring, K can be generated over R by one element

def: a prime ideal P in a commutative ring R is called a G-ideal if R/P is a G-domain

Theorem 19: Let R be a domain with quotient field K. For $0 \neq u \in R$ TFAE:

(1) Any nonzero prime ideal contains u

(2) Any nonzero ideal contains a power of u

(3) $K = R[u^{-1}]$

any G-domain contains such an element u

Theorem 21: If R is an integral domain and X is an indeterminate over R, then $R[X]$ is never a G-domain.

Theorem 27: An ideal I in a ring R is a G-ideal *iff* it is the contraction of a maximal ideal in the polynomial ring $R[X]$.

proof: ($\Rightarrow$) Suppose Q is a G-ideal. Then by Theorem 27, Q is the contraction of a maximal ideal in $(R[x])[y]$, call it M. Thus $M \cap R[X] = Q$. Since $Q \cap R = P$, we have $M \cap R = (M \cap R[X]) \cap R = Q \cap R = P$. So P is the contraction of a maximal ideal and thus is a G-ideal.

Now note that $P \subseteq Q \Rightarrow PR[X] \subseteq Q$. Towards a contradiction, suppose $PR[X] = Q$. Then, $R[X]/Q = R[X]/PR[X] \cong (R/P)[X]$ implies that $(R/P)[X]$ is a G-domain, a contradiction, since a polynomial ring over an integral domain is never a G-domain. It follows that $PR[X] \subsetneq Q$.

($\Longleftarrow$) Now suppose that P is a G-ideal and Q properly contains $PR[X]$. We want to show Q is a G-ideal. Since P is a G-ideal, R/P is a G-domain, so there exists a $\bar{u} \in R/P$ so that $u \in P'$ for every prime ideal P' of R such that $P \subsetneq P'$. Let $Q' \subseteq R[X]$ be prime such that $Q \subsetneq Q'$. Then we have $PR[X] \subsetneq Q \subsetneq Q'$ is a chain of three distinct prime ideals. Note that $PR[X] \cap R = P$ and $Q \cap R = P$, so the contraction of Q' to R must properly contain P. Then $u \in Q' \cap R$, so $u \in Q'$. Note that $u \notin Q$, since $u \in Q \Rightarrow u \in Q \cap R = P$, a contradiction. Thus $\exists\, u \in R[X]$ with $u \notin Q$ but $u \in Q'$ for every prime ideal Q' containing P. Restated, we can find $\bar{u} \in R[X]/Q$ so that u is contained in every prime in $R[X]$ that properly contains Q. So $R[X]/Q$ is a G-domain and it follows that Q is a G-ideal. $\square$

② Let Q be a G-ideal in $R[X_1, \dots, X_n]$ contracting to P in R. Prove: $\operatorname{rank}(Q) \geq n + \operatorname{rank}(P)$.

proof: we proceed by induction. For the base case let $Q \subseteq R[X_1]$ be a G-ideal with $Q \cap R = P$. Then by the previous exercise, P is a G-ideal and $Q \supsetneq P_1R[X_1]$. Then by Theorem 38, we have $\operatorname{rank}(Q) \geq 1 + \operatorname{rank}(P)$.

Now assume that if $Q \subseteq R[X_1, \dots, X_{n-1}]$ is a G-ideal contracting to P in R, then $\operatorname{rank}(Q) \geq n-1 + \operatorname{rank}(P)$.

Let $R' = R[X_1, \dots, X_{n-1}]$ and let $Q \subseteq R[X_1, \dots, X_n] = R'[X_n]$ be a G-ideal contracting to P_n in R. Consider $Q \cap R' = P'$. since Q is a G-ideal, by the previous exercise P' is a G-ideal and $P'R'[X_n] \subsetneq Q$. since $Q \cap R = P$, then $P' \cap R = P_n$. By induction hypothesis, $\operatorname{rank}(P') \geq n-1 + \operatorname{rank}(P)$ and by the base case $\operatorname{rank}(Q) \geq 1 + \operatorname{rank}(P')$. Then we have,

$$\begin{aligned} \operatorname{rank}(Q) &\geq 1 + \operatorname{rank}(P') \\ &\geq 1 + n - 1 + \operatorname{rank}(P) \\ &\geq n + \operatorname{rank}(P). \end{aligned}$$

$\square$