I. Hilbert Series

Def: An S-module M is $\mathbb{N}^n$-graded if $M = \bigoplus_{b \in \mathbb{N}^n} M_b$ and $x^a M_b \subseteq M_{a+b}$. If the vector space dimension $\dim_k(M_a)$ is finite for all $a \in \mathbb{N}^n$, then the *finely graded* or *$\mathbb{N}^n$-graded Hilbert series of M* is given by:

$$H(M; x) = \sum_{a \in \mathbb{N}^n} \dim_k(M_a) \cdot x^a$$

The *$\mathbb{Z}$-graded* or *coarse Hilbert series of M* is obtained by setting each $x_i = t$, i.e. $H(M; t, \dots, t)$.

Remark: finely graded Hilbert series live in the ring of formal power series $\mathbb{Z}[[x]]$. Here, each element $1 - x_i$ is invertible with inverse $\frac{1}{1-x_i} = 1 + x_i + x_i^2 + \cdots$

example: ↳ The Hilbert series of S is given by:

$$H(S; x) = \prod_{i=1}^{n} \frac{1}{1-x_i}$$

rational function

sum of all monomials in S

Note: ↳ $H(S/I; x)$ = sum of all monomials not in I where $I \subseteq S$ is a monomial ideal

↳ $$H(S(-a); x) = \frac{x^a}{\prod_{i=1}^{n}(1-x_i)}$$

just $x^a \cdot H(S; x)$

- free module generated in degree a
- $S(-a) \cong \langle x^a \rangle$ as $\mathbb{N}^n$-graded modules

Def: The *K-polynomial*, denoted $\mathcal{K}(M;x)$, of an $\mathbb{N}^n$-graded S-module M is the numerator of the Hilbert series of M, $H(M,x)$, when it is expressed as a rational function,

$$H(M,x) = \frac{\mathcal{K}(M;x)}{(1-x_1)\cdots(1-x_n)}$$

Def: For $a \in \mathbb{N}^n$, the *support of a*, *$\mathrm{supp}(a)$*, is the set:

$$\mathrm{supp}(a) = \{ i \in \{1,\dots,n\} \mid a_i \neq 0 \}$$

Note: A monomial x^a is not in I_Δ *iff* $x^{\mathrm{supp}(a)}$ is not in I_Δ.

Theorem: The Hilbert series of the Stanley-Reiser Ring is given by: $H(S/I_\Delta, x) = \sum_{\sigma \in \Delta} \prod_{i \in \sigma} \frac{x_i}{1-x_i}$

proof: $H(S/I_\Delta, x) = \sum \{ x^a \mid a \in \mathbb{N}^n \text{ and } \mathrm{supp}(a) \in \Delta \}$

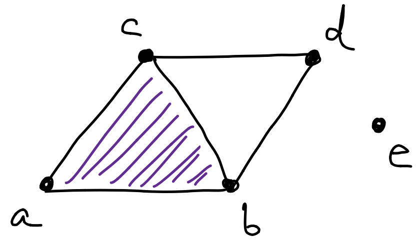
$$= \sum_{\sigma \in \Delta} \sum \{ x^a \mid a \in \mathbb{N}^n \text{ and } \mathrm{supp}(a) = \sigma \}$$

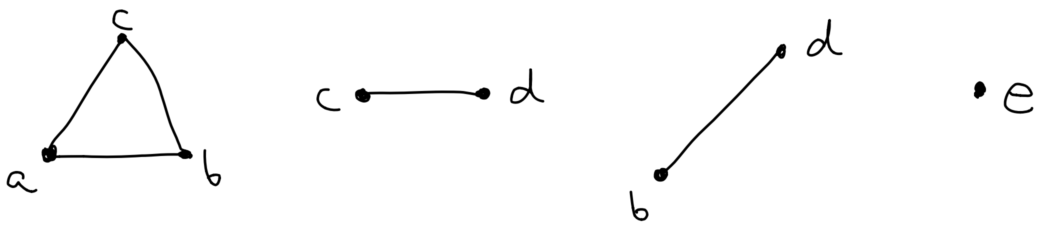
$$= \sum_{\sigma \in \Delta} \prod_{i \in \sigma} \frac{x_i}{1-x_i}$$

$$= \frac{1}{(1-x_1)\cdots(1-x_n)} \cdot \underbrace{\left\{ \sum_{\sigma \in \Delta} \prod_{i \in \sigma} x_i \cdot \prod_{j \notin \sigma} (1-x_j) \right\}}_{\mathcal{K}\text{-polynomial of } S/I_\Delta}$$

II. Examples

example 1

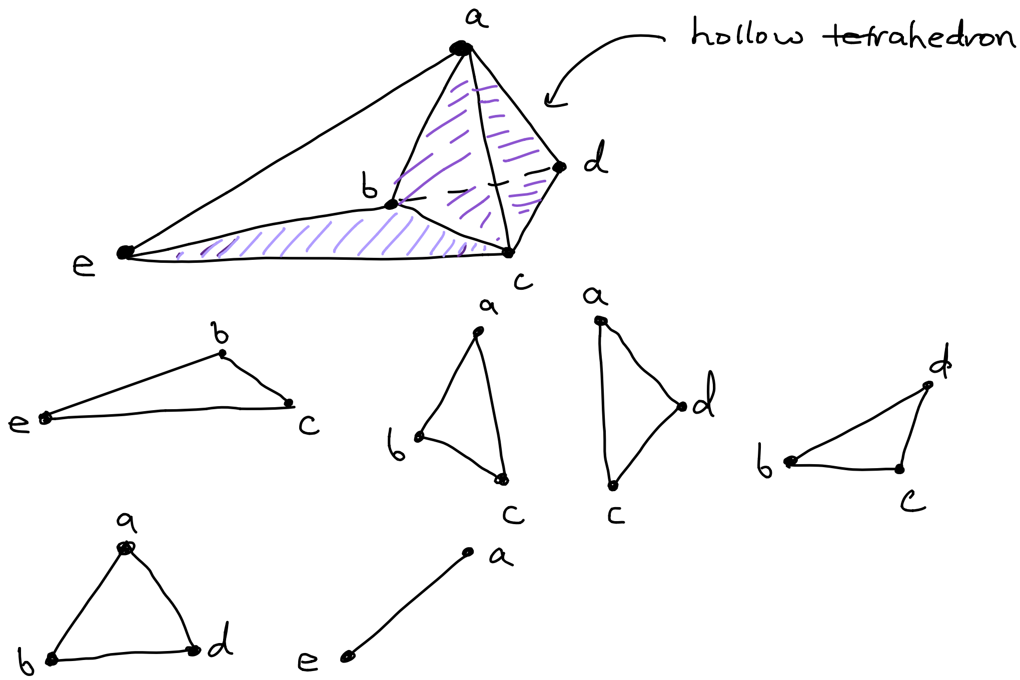




$$I_\Delta = \langle d,e\rangle \cap \langle a,b,e\rangle \cap \langle a,c,e\rangle \cap \langle a,b,c,d\rangle$$
$$= \langle ad, ae, be, ce, de, bcd\rangle$$

$$H(S/I_\Delta, a,b,c,d,e) = 1 + \frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} + \frac{d}{1-d} + \frac{e}{1-e} + \frac{ab}{(1-a)(1-b)}$$
$$+ \frac{ac}{(1-a)(1-c)} + \frac{bc}{(1-b)(1-c)} + \frac{bd}{(1-b)(1-d)} + \frac{cd}{(1-c)(1-d)}$$
$$+ \frac{abc}{(1-a)(1-b)(1-c)}$$

example 2



$$I_\Delta = \langle a,d \rangle \cap \langle d,e \rangle \cap \langle b,e \rangle \cap \langle c,e \rangle \cap \langle b,c,d \rangle$$
$$= \langle de,\ ace,\ abe,\ abcd \rangle$$

$$H(S/I_\Delta, \Lambda^{a,b,c,d,e}) = 1 + \frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} + \frac{d}{1-d} + \frac{e}{1-e} + \frac{ab}{(1-a)(1-b)} + \frac{ac}{(1-a)(1-c)} + \frac{ad}{(1-a)(1-d)}$$

$$+ \frac{ae}{(1-a)(1-e)} + \frac{bc}{(1-b)(1-c)} + \frac{bd}{(1-b)(1-d)} + \frac{be}{(1-b)(1-e)} + \frac{cd}{(1-c)(1-d)} + \frac{ce}{(1-c)(1-e)}$$

$$+ \frac{abc}{(1-a)(1-b)(1-c)} + \frac{abd}{(1-a)(1-b)(1-d)} + \frac{acd}{(1-a)(1-c)(1-d)} + \frac{bcd}{(1-b)(1-c)(1-d)}$$

$$+ \frac{bce}{(1-b)(1-c)(1-e)}$$

K-polynomial

$$= \frac{1 - abcd - abe - ace - de + abce + abde + acde}{(1-a)(1-b)(1-c)(1-d)(1-e)}$$