Title: Characterizing Squarefree Monomial Ideals Using Simplicial complexes

Abstract: A monomial in $k[x_1, \ldots, x_n]$ is a product $x^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ for a vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$. An ideal $I \subseteq k[x_1, \ldots, x_n]$ is called a monomial ideal if it is generated by monomials. We say a monomial is squarefree if every coordinate of $\mathbf{a}$ is 0 or 1, and an ideal is squarefree if it is generated by squarefree monomials. In this talk, we discuss a connection between squarefree monomial ideals and simplicial topology.

We begin with a series of important facts about monomial ideals. Next we provide a brief overview of a simplicial complex Δ on the set of n vertices. Last, we introduce Stanley-Reisner ideals and give a bijective correspondence between squarefree monomial ideals in $k[x_1, \ldots, x_n]$ and a simplicial complex on a set of n vertices.

I. Introduction

Denote the polynomial ring in n indeterminates over a field k by $S = k[x]$ where $x = x_1, \dots, x_n$.

sometimes called exponent vector

Def 1: a *monomial* in S is a product $x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ for a vector $a = (a_1, \dots, a_n) \in \mathbb{N}^n$. An ideal $I \subseteq S$ is called a *monomial ideal* if it is generated by monomials.

nonnegative integers

ex. $I = \langle x^4y^2, x^3y^4, x^2y^5 \rangle \subset k[x,y]$

Remark: a term in $k[x]$ is a polynomial of the form $c\,x^u$ with $c \in k \setminus \{0\}$, if exponent vectors are unique then a polynomial can be written uniquely as the sum of terms

Def 2: a *squarefree monomial* is a monomial x^a where every coordinate of the vector a is 0 or 1. An ideal $I \subseteq S$ is a *squarefree monomial ideal* if it is generated by squarefree monomials.

Squarefree monomial ideals are important in that the information carried by them can be characterized in many ways. The focus of this talk is to focus on one of the most important characterizations, that between squarefree monomial ideals and simplicial complexes.

$\boxed{\text{II. Background}}$

A. Monomial Ideals

Fact 1: As a k-vector space, the polynomial ring S is a direct sum $S = \bigoplus_{a \in \mathbb{N}^n} S_a$ where S_a is the k-span of the monomial x^a.

↳ the polynomials $x_1, \ldots, x_n$ form a vector space over k which has the set of all monomials as a basis

↳ $S_a = k\{x^a\}$ is the vector subspace of S spanned by the monomial x^a, i.e. $S_a = \{\sum_{a \in \mathbb{N}^n} c x^a \mid c \in k\}$

Fact 2: S is an $\mathbb{N}^n$-graded k-algebra

↳ $S = \bigoplus_{a \in \mathbb{N}^n} S_a$ ↳ $S_a \cdot S_b \subseteq S_{a+b}$ subspace spanned by x^{a+b} where $a+b = (a_1+b_1, \ldots, a_n+b_n)$

↳ $S_n = \{\sum_{m \in \mathbb{N}^d} r_m x^m \mid r_m \in R, m_1 + \cdots + m_d = n\}$ where $m = (m_1, \ldots, m_d)$ is the standard grading on a polynomial ring

Fact 3: Monomial ideals are the $\mathbb{N}^n$-graded ideals of S

↳ monomial ideals I can be expressed $I = \bigoplus_{x^a \in I} k\{x\}$

↳ R is a graded subring of S if $R = \sum_n S_n \cap R$ or if for every $f \in R$ all the homogeneous components of f are in R (hom. comp. are monomials)

Fact 4: A monomial x^β is in $I = \langle x^\alpha : \alpha \in A \rangle$ iff x^β is divisible by $x^{\alpha(i)}$ for some $\alpha(i) \in A$

Note: $x^\alpha \mid x^\beta$ iff $\beta(i) \leq \alpha(i) \ \forall i$

proof: $\Rightarrow$ Let $x^\beta \in I$, then $x^\beta = \sum_{i=1}^{s} h_i x^{\alpha(i)}$ where $h_i \in k[x]$ and $\alpha(i) \in A$. If we expand each h_i as a linear combination of monomials, each term on the right side of the equation is divisible by some $x^{\alpha(i)}$. Hence, the left side x^β must have the same property.

$\Leftarrow$ If x^β is a multiple of x^α for some $\alpha \in A$ then $x^\beta \in I$ by definition

Fact 5: A polynomial $f \in I$ iff every term of f lies in I ↑ also why I is a graded subring

proof: $\Rightarrow$ Let $f \in I$, then $f = \sum_{i=1}^{s} h_i x^{\alpha(i)}$ where $h_i \in k[x]$ and $\alpha(i) \in A$. We can express each h_i as a k-linear combination of monomials. If we expand the products we have f is a k-linear combination of terms of the form $x^j x^{\alpha(i)}$ for various $j \in \mathbb{Z}^n_{\geq 0}$ and $\alpha(i) \in A$.

$\Leftarrow$ If every term of f lies in I, it is clear that f lies in I.

Remark: Facts 4 and 5 tell us that a monomial ideal I contains monomials and polynomials f where each monomial of f is in I. In general, if an ideal contains a polynomial it is not necessary for its monomials to be in that ideal unless the ideal is a monomial ideal.

example: Consider $I = \langle xy, y^3 \rangle \leq \mathbb{Q}[x,y]$

↳ $x^2y,\ x^3y,\ x^3y^3,\ y^4 \in I$ (all monomial terms)

↳ $4xy + y^3,\ x^2y + xy^3 - xy^2 + y^3 \in I$ (also contains polynomials)

$(1+x)y^3 + (x-y)xy$

↳ $x - xy,\ y-1,\ y^4 - 1 \notin I$

example: Consider $I = \langle x+y \rangle \leq \mathbb{Q}[x,y]$

↳ $\underset{\text{in } I}{y^3(x+y)} = \underbrace{xy^3 + y^4}_{\text{monomials are not in } I}$

Fact 6: A monomial ideal has a unique minimal set of monomial generators, and this set is finite.

in general, the choice of generators of an ideal is not unique, however, in the context of monomial ideals, the set of monomial generators is unique

Hilbert's basis theorem: Let k be a field, $n \in \mathbb{N}$ and I an ideal in $k[x_1, \dots, x_n]$. Then there exists a finite set $f_1, \dots, f_m$ of polynomials such that $I = \langle f_1, \dots, f_m \rangle$.

tells us every monomial ideal is finitely generated.

Lemma: Every monomial ideal $I \subseteq k[x_1, \dots, x_n]$ has a unique monomial generating set.

← F is the set of all monomials that are not divisible by any other monomial. If $x^a \in I$ then there does not exist any other monomial that divides it in I but itself

proof: Let $F = \{ x^a \in I \mid \forall x^b \in I \setminus \{x^a\} : x^b \nmid x^a \}$. We claim that F generates I. We want to show every monomial $x^c \in I$ is divisible by some element of F. If $x^c \in F$ then $x^c \mid x^c$. If $x^c \notin F$ then there exists $x^{c'} \in I / \{x^c\}$ ← there is some monomial in I other than itself that divides it such that $x^{c'} \mid x^c$. If $x^{c'} \in F$, then we are done. If $x^{c'} \notin F$, then there exists $x^{c''} \in I / \{x^{c'}\}$ such that $x^{c''} \mid x^{c'} \mid x^c$. If $x^{c''} \in F$ we are done, otherwise we continue with this process. It must eventually stop since the integer entries of the exponent vectors become smaller and smaller. Thus, $\exists\, x^a \in F$ s.t. $x^a \mid x^c$. Since no other generator can divide these elements x, F is contained in any monomial generating set for I. It follows that F is minimal and unique. $\blacksquare$

Remark: Fact 6 was proved using Hilbert's basis theorem and the above Lemma. Often times this is shown using Dickson's Lemma which is an application of Hilbert's basis theorem in the monomial case.

Dickson's Lemma: Every monomial ideal $I \subseteq k[x_1, \dots, x_n]$ has a unique finite monomial generating set.

↑ Hilbert's basis theorem in the monomial case

Fact 7: Every monomial ideal can be expressed as a unique irredundant intersection of irreducible monomials. We write an irreducible monomial ideal as $m^a = \langle x_i^{a_i} \mid a_i \geq 1 \rangle$ for $a \in \mathbb{N}^n$. If $\sigma \subseteq \{1, \ldots, n\}$, then $m^\sigma = \langle x_i \mid i \in \sigma \rangle$

Remark: Given $I = \langle m_1, \ldots, m_k \rangle$ and $J = \langle n_1, \ldots, n_r \rangle$ the generating set for $I \cap J = \langle \operatorname{lcm}(m_i, n_j) : 1 \leq i \leq k,\ 1 \leq j \leq r \rangle$ and we remove elements which are non-minimal under divisibility to make a minimal generating set for $I \cap J$ ← nice generating set for intersection of monomial ideals

example: $I = \langle x^3, xy^2, z^2 \rangle$ and $J = \langle x^2y, xyz, yz^3 \rangle$

$$\Rightarrow I \cap J = \langle \operatorname{lcm}(x^3, x^2y), \operatorname{lcm}(x^3, xyz), \operatorname{lcm}(x^3, yz^2), \operatorname{lcm}(xy^2, x^2y), \operatorname{lcm}(xy^2, xyz), \operatorname{lcm}(xy^2, yz), \operatorname{lcm}(z^2, x^2y), \operatorname{lcm}(z^2, xyz), \operatorname{lcm}(z^2, yz^3) \rangle$$

$$\Rightarrow I \cap J = \langle x^3y, x^3yz, x^3yz^3, x^2y^2, xy^2z, x^2yz^2, xyz^2, yz^3 \rangle$$

$$= \langle x^3y, x^2y^2, xy^2z, xyz^2, yz^3 \rangle$$

after removing all nonminimal monomial generators

Remark: with squarefree monomial ideals we can interpret this intersection combinatorically

Remark : Facts 6 and 7 tell us there are two ways to present a monomial ideal:

(1) by its minimal generators

(2) as an intersection of monomial prime ideals

irreducible implies prime in $S = k[\cancel{x}]$

B. Simplicial Complexes

Def 3: An *abstract simplicial complex* Δ on n vertices $\{1, 2, \ldots, n\}$ is a collection of subsets called *faces* or *simplices* closed under taking subsets, i.e. if $\sigma \in \Delta$ is a face and $\tau \subseteq \sigma$, then $\tau \in \Delta$

Def 4: A simplex $\sigma \in \Delta$ of cardinality $|\sigma| = i+1$ has *dimension* i and is called an *i-face* of Δ. The *dimension of Δ, $\dim(\Delta)$*, is the maximum of the dimensions of its faces.

Note:

- If $\Delta = \{\}$ is the void complex, $\dim(\Delta) = -\infty$
- the empty set is the unique dimension -1 face in any simplicial complex Δ
- If $\Delta = \{\emptyset\}$ is the irrelevant complex, $\dim(\Delta) = -1$

Note: a maximal face is called a facet, a simplicial complex is completely determined by its facets

example

ⓐ define the simplicial complex Δ on $[5] := \{1, 2, 3, 4, 5\}$ consisting of all subsets of the sets $\{1, 2, 3\}$, $\{2, 4\}$, $\{3, 4\}$, and $\{5\}$

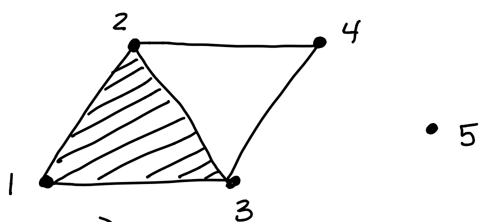
$F_2(\Delta) = \{ \{1,2,3\} \}$

$F_1(\Delta) = \{ \{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\} \}$

$F_0(\Delta) = \{ \{1\}, \{2\}, \{3\}, \{4\}, \{5\} \}$

$F_{-1}(\Delta) = \{ \emptyset \}$

$\dim(\Delta) = 2$



$f = (1, 5, 5, 1)$

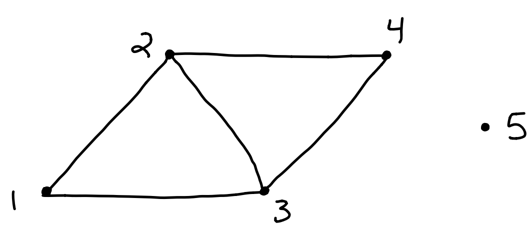
(b) define the simplicial complex Δ on $[5]$ consisting of all subsets on $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$, and $\{5\}$

$F_1(\Delta) = \{ \{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\} \}$

$F_0(\Delta) = \{ \{1\}, \{2\}, \{3\}, \{4\}, \{5\} \}$

$F_{-1}(\Delta) = \{ \emptyset \}$

$\dim(\Delta) = 2$



$f = (1, 5, 5)$

(c) define the simplicial complex Δ on the set $[6]$ consisting of all subsets of $\{1,2,4\}$, $\{2,4,5\}$, $\{2,3\}$, $\{3,5\}$, and

$\{3,6\}$

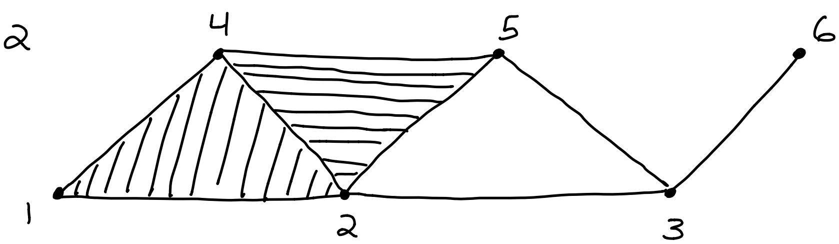
$F_2(\Delta) = \{\{2,4,5\}, \{1,2,4\}\}$

$F_1(\Delta) = \{\{2,4\}, \{2,5\}, \{4,5\}, \{3,5\}, \{1,2\}, \{1,4\}, \{2,3\}, \{3,6\}\}$

$F_0(\Delta) = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$

$F_{-1}(\Delta) = \{\emptyset\}$

$\dim(\Delta) = 2$



$f = (1, 6, 8, 2)$

III. Stanley-Reisner Ideals

Fact 8 simplicial complexes determine squarefree monomial ideals.

↳ identify each subset $\sigma \subseteq \{1,\dots,n\}$ with its squarefree vector in $\{0,1\}^n$ which has a 1 in the i^{th} spot when $i \in \sigma$ and 0 in all other entries. so we can write,

set of n-tuples comprised of 0's and 1's

$$x^\sigma = \prod_{i\in\sigma} x_i$$

example: $\sigma = \{2,4\} \subseteq \{1,2,3,4,5\} \Leftrightarrow a = (0,1,0,1,0)$

$x^\sigma = x_2 x_4$

Def 5: The *Stanley-Reisner ideal* of the simplicial complex Δ is the squarefree monomial ideal

$$I_\Delta = \langle x^\tau \mid \tau \notin \Delta \rangle$$

generated by monomials corresponding to *nonfaces* τ of Δ. The *Stanley-Reisner ring* of Δ is the quotient ring S/I_Δ.

Recall: There are two ways to present a squarefree monomial ideal:

(1) by its minimal generators ← from Fact 6

(2) as an intersection of monomial prime ideals ← from Fact 7

↑ write $m^\tau = \langle x_i \mid i \in \tau \rangle$, usually τ is $\bar{\sigma} = \{1,\dots,n\} \setminus \sigma$ complement of simplex

Theorem 1: The correspondence $\Delta \rightsquigarrow I_\Delta$ constitutes a bijection from simplicial complexes on vertices $\{1,\dots,n\}$ to squarefree monomial ideals inside $S = k[x]$. Furthermore,

simplicial complex

stanley reisner ideal

$$I_\Delta = \bigcap_{\sigma \in \Delta} m^{\bar{\sigma}}$$

Bijection

stanley-reisner ring

Note: the nonzero square free monomials in S/I_Δ correspond to the faces of Δ, $\{x^\sigma \mid \sigma \in \Delta\}$

(pass to S/I_Δ)

the stanley reiser complex Δ associated to I_Δ is the simplicial complex on n vertices with
$\Delta = \{\sigma \subseteq \{1,\dots,n\} \mid \notin \Delta$ for any $x^\tau \in I_\Delta\}$

$\Updownarrow$

squarefree monomial ideal associated to a simplicial complex Δ is
$I_\Delta = \{x^\tau \mid \tau \notin \Delta\}$

monomials correspond to the nonfaces of Δ

$I_\Delta = \bigcap_{\sigma \in \Delta} m^{\bar{\sigma}}$

Let $x^\tau \in I_\Delta$. (then τ is a non-face of Δ)

what does it mean for $x^\tau \in \bigcap_{\sigma \in \Delta} m^{\bar{\sigma}}$?

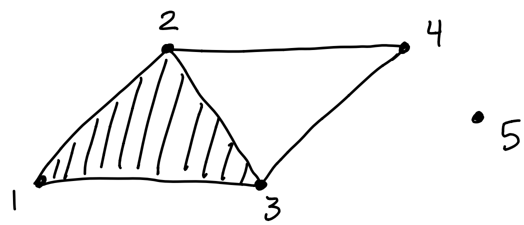
- τ has to share at least one element with $\bar{\sigma}$ for each face $\sigma \in \Delta$, this means for every facet $\bar{\sigma} \in \Delta$, τ has at least one vertex not in $\bar{\sigma}$

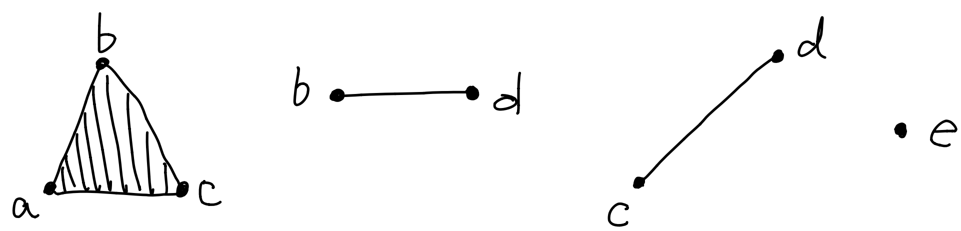
$m^{\bar{\sigma}} = \langle x_i \mid i \in \bar{\sigma} \rangle$

where $\bar{\sigma} = \{1,\dots,n\}/\sigma$

example: We can use the correspondence to compute the Stanley Reisner Ideal of a simplicial complex

(a) Consider the simplicial complex Δ on $[5] := \{1,2,3,4,5\}$ consisting of all subsets of the sets $\{1,2,3\}$, $\{2,4\}$, $\{3,4\}$, and $\{5\}$



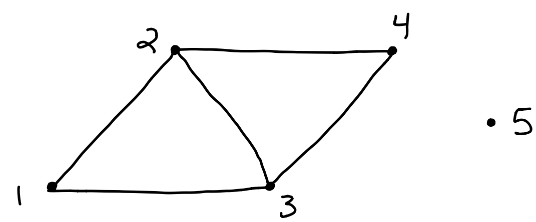


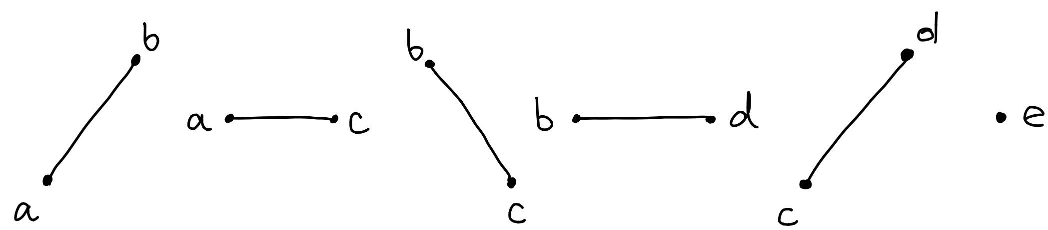
$$
\begin{aligned}
I_\Delta &= \langle d,e\rangle \cap \langle a,c,e\rangle \cap \langle a,b,e\rangle \cap \langle a,b,c,d\rangle \\
&= \langle ad,\ ae,\ bcd,\ be,\ ce,\ de\rangle
\end{aligned}
$$

missing edges and missing face

generators are minimal non-faces of Δ

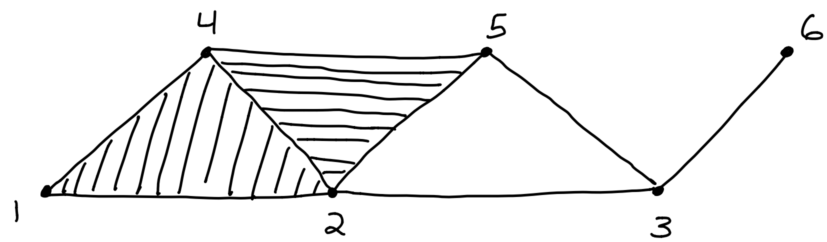
6 Consider the simplicial complex Δ on $[5]$ consisting of all subsets on $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$, and $\{5\}$

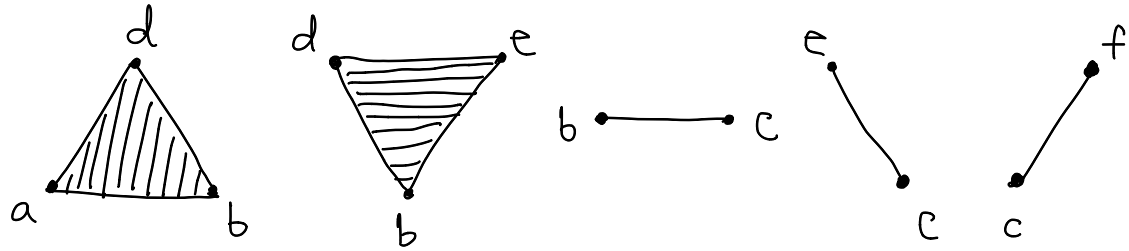




$$I_\Delta = \langle c,d,e\rangle \cap \langle b,d,e\rangle \cap \langle a,d,e\rangle \cap \langle a,c,e\rangle \cap \langle a,b,e\rangle \cap \langle a,b,c,d\rangle$$
$$= \langle ad, ae, abc, be, bcd, ce, de\rangle$$

(c) Consider the simplicial complex Δ on the set $[6]$ consisting of all subsets of $\{1,2,4\}$, $\{2,4,5\}$, $\{2,3\}$, $\{3,5\}$, and





$$I_\Delta = \langle c,e,f\rangle \cap \langle a,c,f\rangle \cap \langle a,d,e,f\rangle \cap \langle a,b,d,f\rangle \cap \langle a,b,d,e\rangle$$
$$= \langle ae, ac, af, bf, bce, cd, df, ef\rangle$$

squarefree monomial ideals are important in that the information carried by them can be characterized in many ways. several of the most important are given in the following theorem.

Theorem 3:

(1) squarefree monomial ideals in $S = K[x]$ where $x = x_1, \ldots, x_n$.

(2) unions of coordinate subspaces in k^n

(3) unions of coordinate subspaces in $\mathbf{P}^{n-1}$

(4) simplicial complexes on $\{1, \ldots, n\} := \{1, 2, \ldots, n\}$

(4) $\Leftrightarrow$ (1) ✓

(1) $\Rightarrow$ (2) : Given a squarefree monomial ideal I, let $V(I) \subset k^n$ be defined by $V(I) := \{a \in k^n : \forall f \in I : f(a) = 0\}$. Then since we can write $I = m^{\sigma_1} \cap \cdots \cap m^{\sigma_r}$ as an intersection of monomial prime ideals we can write $V(I) = V(m^{\sigma_1}) \cup \cdots \cup V(m^{\sigma_r})$ as the union of coordinate subspaces where $V(m^{\sigma})$ is the vector subspace of k^n spanned by the standard basis vectors $\{e_j \mid j \notin \sigma\}$

(2) $\Rightarrow$ (3): $\mathbb{P}^{n-1} = (k^n \setminus \{0\})/k^*$ is a quotient of k^n and The quotient of any nonzero coordinate subspace of k^n is a coordinate subspace of $\mathbb{P}^{n-1}$. Thus, there is a 1-1 correspondence between affine and projective coordinate subspaces where the empty set is a projective coordinate subspace, spanned by the empty set of coordinate points, corresponding to the affine linear space subspace $\{0\}$.

(3) $\Rightarrow$ (4): The simplicial complex on $\{1, \dots, n\}$ corresponding to a union of coordinate subspaces is the one whose faces consist of those sets $\sigma \subseteq \{1, \dots, n\}$ such that $\operatorname{span}(e_i \mid i \in \sigma)$ is contained in the union. It is a simplicial complex because if a subspace V is contained in the union, so is every subspace of V.

IDEA

ideal $I \longleftrightarrow$ affine variety of $I \longleftrightarrow$ projective variety of I

coordinate subspace $\longleftrightarrow$ simplex

Remark: It is not always true that the ideal of polynomials vanishing on a collection of coordinate subspaces is a monomial ideal! This correspondence from $4 \leftrightarrow 3 \leftrightarrow 2 \leftrightarrow 1$ is NOT the Zarski correspondence, it doesn't hold if k is finite.

If k is infinite the Zarski correspondence induces a correspondence between squarefree monomial ideals and coordinates subspaces.