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Research Statement

I. Background

My research focus is in a branch of abstract algebra called commutative ring theory. In particular, I am interested in the factorization properties of commutative rings, especially those with zero-divisors. For integral domains, factorization theory has been well established. There are standard definitions for the building blocks of such rings, such as irreducible and prime elements, as well as certain ring theoretic properties, like unique factorization. When discussing factorization in commutative rings with zero divisors however, things become more complicated. For example, the presence of zero-divisors has led to different definitions of irreducible and associate elements by several authors, and different factorization techniques have led to several types of unique factorization rings.

My research focuses on how factorization properties behave with respect to certain extensions. Of particular interest is the extension of a commutative ring with identity R to the polynomial ring $R[X]$: if one of these rings has a certain factorization property, does the other? If the answer is no, we would like to have nice counterexamples, and polynomial rings often give much simpler examples than you might see for arbitrary commutative rings.

II. Current Research

My current research has the goal of creating an overarching theory for factorization in polynomial rings with zero divisors. I aim to do this by focusing on their behavior with respect to many important factorization properties such as unique factorization, bounded factorization, finite factorization, the ascending chain condition on principal ideals (ACCP), and atomicity. To this end I have focused primarily on unique factorization.

For an integral domain R, it is well known that $R[X]$ is a unique factorization domain if and only if R is a unique factorization domain. This result does not hold if we generalize to polynomial rings with zero divisors. In fact, with zero divisors we have multiple notions of a unique factorization ring. Thus, we can consider when a polynomial ring is a unique factorization ring (UFR) in the sense of Bouvier and Galovich and when it is a UFR in the sense of Fletcher.

Though both Galovich and Fletcher established structure theorems to characterize unique factorization in arbitrary rings, the inherent structure of polynomial rings affords surprisingly straightforward proofs of standard results without the use of such powerful machinery. Specifically, I was able to show an elementary proof that $R[X]$ is a UFR in the sense of Bouvier and Galovich if and only if R is a UFD. I also showed that for an arbitrary commutative ring R, the following are equivalent: (1) $R[X]$ is a UFR in the sense of Fletcher (2) each (regular) nonzero nonunit has unique factorization into irreducible elements (3) each (regular) nonunit elements is a product of principal primes (4) R is a finite direct product of UFDS.

There are still other unique factorization properties to be considered. For example, I would like to characterize when a polynomial ring is a strongly reduced UFR. I have also briefly looked into non-unique factorization in polynomial rings. For example, X^n has non-

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unique factorization in $\mathbb{Z}_4[X]$ and I was able to give a formula for the length of factorizations of X^n in $\mathbb{Z}_4[X]$ that could be extended to $\mathbb{Z}_{p^n}[X]$. I plan to continue investigating with X^n has unique factorization in $R[X]$ where R is a commutative ring with zero divisors.

I am also interested in counterexamples. W. Heinzer and D. Lantz [10] were able to show by a counterexample that if a commutative ring R has ACCP, then $R[X]$ does not have to satisfy ACCP. If I can show that their example is also not atomic, we will have a nice counterexample that R atomic does not imply $R[X]$ is atomic. Though many counterexamples exist in the domain case, they are often quite complicated and it is of interest to see if we can find new ones using polynomial rings.

III. Future Work

Moving forward I would like to continue exploring topics related to factorization in commutative rings with zero divisors. With respect to polynomial rings, given any property for a commutative ring R, we can always ask under what conditions does that property extend to $R[X]$? In my previous work, much time was spent on characterizing unique factorization in polynomial rings. Other properties to consider include finite factorization, bounded factorization, ACCP, and atomicity. In addition, it is of interest to consider the U-factorization method introduced by C.R. Fletcher, as it relates to polynomial rings. For a commutative ring R, we can then consider when $R[X]$ is a U-UFR, U-BFR, U-HFR, U-FFR, etc.

With respect to general commutative algebra, I would like to spend some time with a book published last year entitled *Multiplicative Ideal Theory and Factorization Theory: Commutative and Non-commutative Perspectives*. This collection of papers gives an expository view of topics concerning multiplicative ideals, factorization, algebraic geometry, and number theory from both a commutative and non-commutative perspective. There are many open problems posed in this volume and I am anxious to spend time using this as a guide to broaden the scope of my long term research in commutative algebra beyond factorization in polynomial rings.

I am also interested in investigating the connections between my research and graph theory. In 1998, I. Beck wrote *The Coloring of Commutative Rings* [12] that established a powerful connection between commutative ring theory and graph theory. From this emerged the notion of a zero-divisor graph, denoted $\Gamma(R)$, which is a graph whose vertices are the nonzero zero-divisors of commutative ring R. In $\Gamma(R)$, two vertices a and b are connected by an edge, denoted $a - b$, if $ab = 0$. These zero-divisor graphs can be used to recover information about the factorization properties of a given commutative ring and thus are of growing interest in the field of commutative ring theory.

This particular area of study lends itself very well to undergraduate research. In fact, M. Axtell and J. Stickles [14] gave six project ideas of various lengths on zero-divisor graphs suitable for undergraduates who have completed an introductory abstract algebra course. I look forward to using these ideas as starting points for new projects that can relate zero-divisor graphs to my research with polynomial rings.

Lastly, in my recent research, I have looked at non-unique factorization of X^n in $\mathbb{Z}_{p^n}[X]$, with $\mathbb{Z}_4[X]$ as a particular example. Factorization properties of elements in polynomial rings over finite rings like $\mathbb{Z}_4[X]$ are quite important in coding theory. In fact, a recent direction of coding theory has been to study linear codes over the alphabet $\mathbb{Z}_4$. There are many such linkages of factorization theory to applied algebra. In the upcoming years, I would like to explore factorizations of polynomials over finite rings and their applications to coding theory, cryptography, and computational algebra with the primary goal of finding meaningful student projects with a more applied flavor.

IV. Teaching and Research

I believe the practice of integrating research into teaching helps to keep courses fresh and relevant for students. Providing insight into why accomplished mathematicians and industry professionals are interested in a certain topic is beneficial to those who choose to pursue mathematics and those who are just taking required courses. It is also important to establish connections between different branches of mathematics. Many ideas from my own research and interests can be used directly in my teaching. I have included several below.

1. Explore Important Ring Theoretic Properties through the Lens of Prime Ideals. For example, a commutative ring R is Noetherian if and only if all of its prime ideals are finitely generated.

2. Explore Properties in Integral Domains weaker than Unique Factorization such as Bounded Factorization Domains, Half-Factorial Domains, Finite Factorization Domains, idf-domains, domains that satisfy ACCP, and atomic domains.

3. Characterize Units, Zero-Divisors, Idempotents, and Nilpotent Elements in Polynomial Rings $R[X]$ where R is an arbitrary commutative ring with identity.

4. Explore a topological connection to ring theory through the Zariski topology on the prime spectrum of a commutative ring, Spec R.

These topics, particularly 1, 3, and 4, could greatly enhance the richness of an undergraduate course in abstract algebra as special assignments. The second topic would be a great way to emphasize why Unique Factorization is so important, with further explorations of weaker factorizations properties left to a reading course, topics course, or independent project. The fourth would be a great guiding question for an in depth exploration of properties of radical ideals, and could also be a particular talk given in a seminar series on the connection between commutative rings and algebraic geometry.

My teaching is also influenced by the pedagogical training I have received while completing the Graduate Certificate in College Teaching and the Graduate Certificate in Online Teaching at the University of Iowa. As a result, I have developed a deep interest in blending learning and inquiry based learning, the latter I have observed in a teaching practicum in discrete mathematics and the former I have explored by partially flipping a trigonometry course as a graduate student instructor of record. I am also completing a capstone project that incorporates both IBL and online learning for an eight week mini-course that serves as a brief introduction to ring theory. I believe these two techniques coupled together can be very powerful in facilitating student independence, an engaging classroom, and deeper learning experiences for students.

V. Undergraduate Research

In previous sections I have alluded to projects I am interested in doing with undergraduate students related to graph theory and also in applications of ring theory that show up in coding theory and cryptography. Below I describe in greater detail specific projects related to my past and present research I could work on with students as I delve deeper into these aforementioned topics. I have also included project ideas related to mathematics education.

Pure Mathematics

1. Complex Sum of Divisors Function: The complex sum of divisors function, denoted by $\sigma^*(\eta)$ is given by $\sigma^*(\eta) = \prod \frac{\pi_i^{k_i+1} - 1}{\pi_i - 1}$ where η is a Gaussian integer such that $\eta = \epsilon \prod \pi_i^{k_i}$ with ϵ a unit and each π_i in the first quadrant. This function was introduced by R. Spira as a natural extension of the standard sum of divisors function on the real numbers to the Gaussian integers. Using it, we can study how many properties that are characterized by the sum of divisors function extend to the Gaussian integers.

   For example, two numbers n_1 and n_2 are said to be amicable if $\sigma(n_1) = n_2$ and $\sigma(n_2) = n_1$. An aliquot sequence is a sequence of positive integers in which each term is the sum of the proper divisors of the previous term. Thus if $s(n) = \sigma(n) - n$, the sequence $s^0(n) = n, s^1(n) = s(n), s^2(n) = s(s(n)) \ldots$, is an aliquot sequence. If a sequence reaches a cycle of length two those two numbers are an amicable pair. Sociable numbers are cycles of length greater than two. A perfect number has a repeating aliquot cycle of 1.

   All of the above properties can be explored using the complex sum of divisors function. The research I did on Gaussian Amicable Pairs in my Master thesis lends itself well to numerous extensions suitable for undergraduate projects. Specifically, we can investigate the following:

   - How can we define Gaussian superperfect numbers? Gaussian norm-superperfect numbers? Do they exist? If so, are there real superperfect numbers that also carry over as Gaussian superperfect numbers? How can we characterize the different types of Gaussian superperfect numbers?

- Do there exist Gaussian aliquot sequences? How can we realize such sequences with a directed graph? What do cycles and loops in such a graph represent?
- Explore relationship between the characterisitcs of Mersenne Primes and Gaussian Mersenne Primes.
- Try to find new Gaussian amicable pairs of a certain type.

2. U-factorization: For a commutative ring R, if $r = a_1 \ldots a_n b_1 \ldots b_m$ is a factorization for a nonunit r, then we say $r = a_1 a_2 \ldots a_n [b_1 b_2 \ldots b_m]$ is a U-factorization of r if (1) $a_i(b_1 \ldots b_m) = (b_1 \ldots b_m)$ for $i = 1, \ldots, n$ and (2) $b_j(b_1 \cdots \hat{b_j} \cdots b_m) \neq (b_1 b_2 \cdots \hat{b_j} \cdots b_m)$ where $\hat{b_j}$ is the deletion of b_j from the product. We call the b_js essential divisors and the a_is inessential divisors of the U-factorization of r. If each a_i and b_j is irreducible then the U-factorization is a U-decomposition of r.

    - Survey U-factorization in arbitrary commutative rings and then extend U-factorization to polynomial rings with zero divisors by looking at when $R[X]$ is a U-UFR, U-BFR, U-HFR, etc.
    - A τ_n-factorization of a nonzero nonunit $a \in \mathbb{Z}$ is a factorization of the form $a = \lambda \cdot a_1 \cdots a_t$ where $t \geq 1$, $\lambda = 1$ or -1, and each a_i satisfies $a_i \equiv a_j \mod n$ for $i, j \in \{1, \ldots, t\}$. Explore connections between U-factorization and τ_n-factorization.

Mathematics Education

3. Online Teaching: Design a short course, seminar, or training intended to be taken online. Consider participation, communication, both synchronous and asynchronous, and assessment in such a setting. Read literature on best practices for course design in an online setting. In the process, look at new trends in the utilization of learning management systems in K-12 and/or post-secondary education. Also consider the pros and cons of blended learning in mathematics education and technologies involved in this instructional approach.

4. Universal Design for Learning: UDL is a framework for course design that encourages the creation of materials that consider accessibility in many different ways. Students will use this framework to create and critique documents for a particular course they could teach in the future. The focus could include sample syllabi, assessments, course projects, or lesson plans. Students should also survey assistive technologies used both in K-12 and post-secondary education.

5. Inquiry Based Learning: Assist in the creation or modification of a course workbook for an undergraduate math course taught using IBL. Help provide feedback on style and problems and help develop one assignment or section of the workbook as an introduction to teaching with IBL in higher education.

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