FACTORIZATION IN POLYNOMIAL RINGS WITH ZERO DIVISORS

by

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CERTIFICATE OF APPROVAL

PH.D. THESIS

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ABSTRACT

In this thesis, we investigate factorization in polynomial rings with zero divisors. Of particular interest is how certain factorization properties behave with respect to the polynomial extension $R[X]$ where R is an arbitrary commutative ring. For example, if R is an integral domain, it is well knowm that R is a unique factorization domain if and only if $R[X]$ is a unique factorization domain. However, if R is a unique factorization ring that is not an integral domain, $R[X]$ is not necessarily a unique factorization ring. In fact, many factorization properties do not extend from a ring R to $R[X]$ if we take R to be a commutative ring with zero divisors. If R is atomic or has ACCP, for example, $R[X]$ does not necessarily inherit these properties.

The central result of this work is the characterization of when a polynomial ring over an arbitrary commutative ring has unique factorization. A characterization is given for when $R[X]$ is a unique factorization ring of the types defined by Bouvier, Galovich, and Fletcher. The technique of reduced factorizations is considered and a characterization of when $R[X]$ is a reduced and a μ-reduced unique factorization ring is given. Characterizations of when $R[X]$ is factorial, an (α, β)-unique factorization ring, and a weak unique factorization ring are also given. Additionally, attention is paid to associate relations, atomicity, and ACCP in $R[X]$. The notion of an indecomposable polynomial is generalized to a polynomial ring with zero divisors and several types of indecomposable polynomials are defined based on different associate conditions.

PUBLIC ABSTRACT

Factorization theory is concerned with the decomposition of mathematical objects. Such an object could be a polynomial, a number in the set of integers, or more generally an element in a ring. A classic example of a ring is the set of integers. If we take any two integers, for example 2 and 3, we know that $2 \cdot 3 = 3 \cdot 2$, which shows that multiplication is commutative. Thus, the integers are a commutative ring. Also, if we take any two integers, call them a and b, and their product $a \cdot b = 0$, we know that a or b must be 0. Any ring that possesses this property is called an integral domain. If there exist two nonzero elements, however, whose product is zero we call such elements zero divisors. This thesis focuses on factorization in commutative rings with zero divisors.

In this work we extend the theory of factorization in commutative rings to polynomial rings with zero divisors. For a commutative ring R with identity and its polynomial extension $R[X]$ the following questions are considered: if one of these rings has a certain factorization property, does the other? If not, what conditions must be in place for the answer to be yes? If there are no suitable conditions, are there counterexamples that demonstrate a polynomial ring can possess one factorization property and not another? Examples are given with respect to the properties of atomicity and ACCP. The central result is a comprehensive characterization of when $R[X]$ is a unique factorization ring.

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CHAPTER 1
INTRODUCTION

Factorization in integral domains has been well established. Many authors have studied various factorization properties. In [3], Anderson, Anderson, and Zafrullah, give an important survey of factorization in integral domains. They focus on many properties weaker than unique factorization which includes half-factorial domains, bounded factorization domains, finite factorization domains, irreducible divisor finite domains, domains which satisfy the ascending chain condition on principal ideals, and atomic domains. The theory of factorization in integral domains can be generalized to commutative rings with zero divisors. In this setting, the presence of zero divisors has led to different definitions of irreducible and associate elements by various authors. In [12] Anderson and Valdes-Leon give a survey of how factorization in commutative with zero divisors is similar to and different than factorization in integral domains. This article made the work of many authors uniform, and serves as a comprehensive reference for factorization in arbitrary commutative rings. This thesis draws heavily on the aforementioned articles.

The purpose of this dissertation is to extend the study factorization in commutative rings with zero divisors to polynomial rings. Given a commutative ring R with identity and its polynomial extension $R[X]$ we consider the following questions: if one of these rings has a certain factorization property does the other? For example, if R is an integral domain, it is well known that R is a unique factorization domain if and only if $R[X]$ is a unique factorization domain. This result does not hold if we assume

R to be an arbitrary commutative ring with zero divisors. In fact, many properties do not extend from a ring to its polynomial ring under these assumptions. If R is atomic or has ACCP, for example, $R[X]$ does not necessarily inherit these properties. In this work we will focus on several types of unique factorization rings with zero divisors, the central result being a characterization of when a polynomial ring over an arbitrary commutative ring has unique factorization with respect to many different factorization techniques.

We begin in Chapter two with a brief review of factorization theory. First irreducible and associated elements are defined in the domain case, and then a number of factorization properties weaker than unique factorization are introduced. Atomic domains, domains that satisfy ACCP, half-factorial domains, bounded factorization domains, finite factorization domains and irreducible-divisor finite domains are defined. Next, this theory is generalized to commutative rings with zero divisors. Associate relations are defined and each type of associate relation leads to a different type of irreducible element which in turn lead to different types of atomicity. From this, the factorization properties in the integral domain case are extended to arbitrary commutative rings with zero divisors and unique factorization rings, half-factorial rings, bounded factorization rings, finite factorization rings, weak finite factorization rings, and irreducible-divisor finite rings are defined.

Next we establish elementary facts about polynomial rings in Chapter 3. Distinguished elements in a polynomial ring such as zero divisors, units, idempotents, and nilpotents are characterized. Associate relations in a polynomial ring are discussed, in particular they are used to give a characterization of when a polynomial ring is présimplifiable. The different types of irreducible elements are discussed in the context of polynomial rings and it is shown that in $R[X]$, f is very strongly irreducible if and only if it is m-irreducible. In the last section of this chapter many examples are given of rings that satisfy various factorization properties. In particular the method of idealization is used to give an example of a ring R that is always atomic and is such that ACCP and the bounded factorization property extend to the polynomial ring.

Chapter 4 has the goal of characterizating when a polynomial ring is a unique factorization ring. We begin by asking the question, when does the indeterminate X have unique factorization in $R[X]$? It is shown that X has unique factorization into the product of n atoms if and only if R is the finite direct product of n indecomposable rings. An example is given to show that X^n does not have to have unique factorization into atoms in $R[X]$ where $R = \mathbb{Z}_4$, but that we can say something about the lengths of its factorizations into atoms. Next, the question is asked, when is $R[X]$ a unique factorization ring? The answer depends on the type of unique factorization ring being considered. A characterization is given for when $R[X]$ is a unique factorization ring of the types defined by Bouvier, Galovich, and Fletcher. (α, β)-unique factorization rings are considered along with factorial rings. Both μ-reduced and reduced unique factorization rings are characterized for $R[X]$. Lastly, a characterization is given for when $R[X]$ is a weak unique factorization ring and there is a discussion of weakly prime elements in a polynomial ring.

The last chapter focuses on generalizing indecomposable polynomials to commutative rings with zero divisors. Equivalent conditions for indecomposable polynomials in an integral domain are given, and these are generalized to what we call a regularly indecomposable polynomial. Some results on regularly indecomposable and regularly decomposable polynomials are given. Then associate conditions are used to define indecomposable, strongly indecomposable, and very strongly indecomposable polynomials in $R[X]$. Examples are given that distinguish the different types of indecomposables. Lastly, it is shown that the indeterminate X is indecomposable if and only if X is irreducible if and only if R is indecomposable while X is regularly indecomposable if and only if X is irreducible and R is a reduced ring.

CHAPTER 2
BRIEF REVIEW OF FACTORIZATION

2.1 Factorization in Integral Domains

We begin our brief discussion on factorization in an integral domain R with irreducible elements. A nonzero nonunit element $a \in R$ is said to be *irreducible* or an *atom* if $a = bc$ implies $b \in U(R)$ or $c \in U(R)$ where $U(R)$ denotes the group of units of R. Two elements $a, b \in R$ are said to be *associated*, denoted $a \sim b$, if $a \mid b$ and $b \mid a$, i.e., $(a) = (b)$. Note that $a \sim b$ if and only if $b = ua$ for some $u \in U(R)$. We then have that the following are equivalent:

1. a is irreducible,
2. $a = bc$ implies $a \sim b$ or $a \sim c$, and
3. (a) is maximal in the set of proper principal ideals of R.

A nonzero element $p \in R$ is said to be *prime* if $p \mid ab \implies p \mid a$ or $p \mid b$ for $a, b \in R$. Every prime element in a ring R is irreducible but not every irreducible element must be prime.

An integral domain R is said to be a *unique factorization domain*, *UFD*, if any nonzero nonunit in R can be written as the product of irreducible elements uniquely up to order and associates. There are a number of factorization properties in R weaker than unique factorization. The weakest property that we will discuss is atomic. An integral domain R is *atomic* if each nonzero nonunit in R can be written as a finite product of atoms, i.e., irreducible elements. R is said to satisfy the *ascending chain condition on principal ideals*, *ACCP*, if there does not exist an infinitely strictly ascending chain of principal ideals of R.

We say that R is a *bounded factorization domain*, *BFD*, if R is atomic and for each nonzero nonunit in R there exists a bound on the length of its factorizations into products of irreducibles. We say R is a *half-factorial domain*, *HFD*, if R is atomic and each factorization of a nonzero nonunit of R into a product of irreducibles has the same length. A domain R is an *irreducible-divisor finite domain*, *idf-domain*, if each nonzero element has at most a finite number of nonassociate irreducible divisors. An *atomic idf-domain* is an *idf*-domain that is atomic. A domain where every nonzero nonunit has a finite number of factorizations up to order and associates is called a *finite factorization domain*, *FFD*. Note that a FFD is an atomic *idf*-domain.

Anderson, Anderson, and Zafrullah give a nice survey of factorization in integral domains in [3]. In this paper they give the following series of implications based on the above definitions, none of which can be reversed.

$$\begin{tikzcd}
 & HFD \arrow[dr, Rightarrow] & & & \\
UFD \arrow[ur, Rightarrow] \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & FFD \arrow[r, Rightarrow] \arrow[d, Rightarrow] & BFD \arrow[r, Rightarrow] & ACCP \arrow[r, Rightarrow] & atomic \\
 & idf\text{-}domain & & &
\end{tikzcd}$$

2.2 Factorization in Commutative Rings with Zero Divisors

In a commutative ring R with zero divisors, as in the domain case, we say that a and b are *associated*, denoted $a \sim b$, if $a \mid b$ and $b \mid a$, i.e., $(a) = (b)$. However in a general commutative ring we have other associate relations to consider. Many different authors have taken different definitions for different associate conditions. Throughout we use the associate conditions defined in [12] by Anderson and Valdes-Leon. Then, two elements a and b in R are *strongly associated*, denoted $a \approx b$, if $a = ub$ for some $u \in U(R)$. We say that a and b are *very strongly associated*, denoted $a \cong b$, if (1) $a \sim b$ and (2) $a = b = 0$ or $a \neq 0$ and $a = rb \implies r \in U(R)$.

Note that the relations $\sim$ and $\approx$ are equivalence relations, but $\cong$ does not have to be. For example, if we consider an idempotent $e \neq 0, 1$ then we have that $(e) = (e)$ so $e \sim e$. However $e = e \cdot e$ and $e \notin U(R)$, by definition, so $e \ncong e$. The case where $a \cong a$ for every $a \in R$ has a special name due to Bouvier. We say a ring R is *présimplifiable* if $x = xy \implies x = 0$ or $y \in U(R)$, i.e., $x \cong x$ for all $x \in R$. A simple example of a présimplifiable ring is an integral domain. If $x = xy$ with $x \neq 0$, then left cancellation gives that $1 = y$.

Each of the associate relations leads to a different type of irreducibility. Again, we take our different types of irreducibles to be those defined by Anderson and Valdes-Leon in [12]. A nonunit element $a \in R$ is *irreducible* if $a = bc \implies a \sim b$ or $a \sim c$. We say $a \in R$ is *strongly irreducible* if $a = bc \implies a \approx b$ or $a \approx c$. Finally, $a \in R$ is *very strong irreducible* if $a = bc \implies a \cong b$ or $a \cong c$. If the ideal generated by a, (a), is maximal in the set of proper principal ideals of R, then a is said to be *m-irreducible*.

Lastly, $p \in R$ if prime is $p \mid ab \implies p \mid a$ or $p \mid b$.

In the domain case, for a nonzero element all of the above types of irreducibility coincide. In a general commutative ring that allows zero divisors we have the following implications for nonzero elements, none of which can be reversed.

$$
\begin{array}{ccccccc}
 & & & & & & \text{prime} \\
 & & & & & & \Downarrow \\
\text{very strongly irreducible} & \Longrightarrow & m\text{-irreducible} & \Longrightarrow & \text{strongly irreducible} & \Longrightarrow & \text{irreducible}
\end{array}
$$

In [12], each form of irreducibility leads to a different form of atomicity. As in the domain case, a commutative ring R is *atomic* if every nonzero nonunit can be written as a finite product of irreducibles (atoms). A ring R is *strongly atomic* if every nonzero nonunit can be written as a finite product of strongly irreducible elements. We say a ring is *very strongly atomic* if every nonzero nonunit can be written as a finite product of very strongly irreducible elements. If every nonzero nonunit can be written as a product of m-irreducible elements, then R is *m-atomic*. Lastly, if every nonzero nonunit can be written as a finite product of primes, then R is *p-atomic*. We have the following implications for different types of atomicity, none of which can be reversed.

$$
\begin{array}{ccccccc}
 & & & & p\text{-atomic} & \Longrightarrow & \text{ACCP} \\
 & & & & \Downarrow & & \Downarrow \\
\text{very strongly atomic} & \Longrightarrow & m\text{-atomic} & \Longrightarrow & \text{strongly atomic} & \Longrightarrow & \text{atomic}
\end{array}
$$

To define a unique factorization ring (UFR) takes some work. First, suppose we have two given factorizations of a nonzero nonunit $a \in R$, with $a = a_1 \cdots a_n = b_1 \cdots b_m$ where a_i, b_j are nonunits. Then these factorizations are said to be *isomorphic*, (respectively *strongly isormorphic, very strongly isomorphic,*) if $n = m$ and there exists $\sigma \in S_n$ such that $a_i \sim b_{\sigma(i)}$ (respectively $a_i \approx b_{\sigma(i)}$, $a_i \cong b_{\sigma(i)}$). Next, R is said to be an (α, β)-*unique factorization ring* if $\alpha \in \{$atomic, strongly atomic, very strongly atomic, m-atomic, p-atomic$\}$ and $\beta \in \{$isomorphic, strongly isomorphic, very strongly isomorphic$\}$ and (1) R is α and (2) any two factorizations of a nonzero nonunit $a \in R$ into irreducibles of the type that defines α are β.

It turns out that all the forms of (α, β)-unique factorization are equivalent for $\alpha \in \{$atomic, strongly atomic, very strongly atomic, m-atomic$\}$ and $\beta \in \{$isomorphic, strongly isomorphic, very strongly isomorphic$\}$. This is because R is présimplifiable for any of these choices of α and β. To see this note that if we have $x = xy$, with $x \neq 0$ and $y \notin U(R)$, we can factor $x = a_1 \cdots a_n$, $y = b_1 \cdots b_m$ into irreducibles of the appropriate type α. Then $x = xy$ implies $a_1 \cdots a_n = a_1 \cdots a_n b_1 \cdots b_m$ which gives two factorizations of x into irreducibles of type α that cannot be β. Thus we say that R is a *unique factorization ring*, *UFR*, if R is an (α, β)-unique factorization ring for some (and hence all) (α, β) except for if α is p-atomic [12, Definition 4.3]. We discuss unique factorization more thoroughly in Section 4.2. In particular, Theorem 4.2.1 gives equivalent descriptions for a Bouvier-Galovich UFR. Anderson and Valdes-Leon relate these equivalent conditions of a Bouvier-Galovich UFR to an (α, β)-UFR in [12, Theorem 4.4]. Other types of unique factorization rings will also be discussed.

The other properties weaker than unique factorization in the domain case also extend to general commutative rings with zero divisors. A ring R is a *half factorial ring*, *HFR*, if R is atomic and for any nonzero nonunit $a \in R$, $a = a_1 \cdots a_n = b_1 \cdots b_m$ with a_i, b_j irreducible implies $n = m$. R is called a *bounded factorization ring*, *BFR*, if for every nonzero nonunit $a \in R$ there exists a natural number $N(a)$ such that for any factorization $a = a_1 \cdots a_n$, where each a_i is a nonunit, then $n \leq N(a)$. A *finite factorization ring*, *FFR*, is a ring R where every nonzero nonunit has only a finite number of factorizations up to order and associates. R is said to be a *weak finite factorization ring*, *WFFR*, if every nonzero nonunit has a finite number of nonassociate divisors. Lastly, we say that R is an *idf-ring* if every nonzero nonunit has at most a finite number of nonassociate irreducible divisors.

Anderson and Valdes-Leon give a nice survey of factorization in commutative rings with zero divisors in [12]. In this paper they give the following series of implications in a commutative ring R based on the above definitions, none of which can be reversed.

$$
\begin{tikzcd}
 & HFR \arrow[dr, Rightarrow] & & & \\
UFR \arrow[ur, Rightarrow] \arrow[r, Rightarrow] \arrow[ddr, Rightarrow] & FFR \arrow[r, Rightarrow] \arrow[d, Rightarrow] & BFR \arrow[r, Rightarrow] & ACCP \arrow[r, Rightarrow] & atomic \\
 & WFFR \arrow[d, Rightarrow] & & & \\
 & idf\text{-}ring & & &
\end{tikzcd}
$$

CHAPTER 3
ELEMENTARY FACTS ABOUT POLYNOMIAL RINGS

3.1 Structure of Zero Divisors, Units, Idempotents, and Nilpotents

Before discussing factorization in polynomial rings it is necessary to establish the basic structure of several types of elements in such rings. In this section we focus on units, nilpotents, zero divisors, and idempotents. First we define these elements for an arbitrary ring, and then we give useful characterizations of these elements in a polynomial ring. These well known characterizations will be used frequently in later sections.

We begin with units in $R[X]$. Now, we have that an element $f(X) \in R[X]$ is a *unit* if there exists a $g(X) \in R[X]$ such that $f(X)g(X) = 1$. However, finding such a g is not always clear or convenient. The following theorem gives a very useful way to determine if an element in a polynomial ring is a unit. Recall that an element $x \in R$ is *nilpotent* if $x^n = 0$ for some $n \geq 1$. The collection of all nilpotent elements in a ring R is called the *nilradical*, denoted $Nil(R)$. It is well known that the nilradical of R can be characterized as the intersection of all prime ideals in R.

Theorem 3.1.1. Let $f \in R[X]$ be given by $f = a_0 + a_1X + \cdots + a_nX^n$. Then f is a unit if and only if $a_0 \in U(R)$ and $a_i \in Nil(R)$ for $i = 1, \ldots, n$.

Proof. ($\Longrightarrow$) Let $P \subset R$ be prime. Let $P[X]$ be its extension to $R[X]$, then $P[X]$ is prime. Then $R[X]/P[X] \cong (R/P)[X]$. Let $f(X) = a_0 + a_1X + \cdots + a_nX^n$ be a unit in $R[X]$. Consider $\overline{f} \in (R/P)[X]$. Since P is prime, R/P is an integral domain, so

$(R/P)[X]$ is an integral domain. So $\overline{f}$ is a unit in $(R/P)[X]$ which implies $deg\overline{f} = 0$ and $\overline{a_0}$ is a unit in $(R/P)[X]$. Now $\overline{a_1}, \ldots, \overline{a_n} = \overline{0}$ in $R/P \implies a_1, \ldots, a_n \in P$. Since this will hold for any prime P we have $a_1, \ldots, a_n \in Nil(R)$. Now we have $a_1X, a_2X^2, \ldots, a_nX^n$ nilpotent which implies $a_1X + a_2X^2 + \cdots + a_nX^n$ is nilpotent. Then $a_0 = f - (a_1X + \cdots + a_nX^n)$ is a unit.

($\Longleftarrow$) Suppose $a_0 \in U(R)$ and $a_i \in Nil(R)$ for $i = 1, \ldots, n$. Then $a_1X, \ldots, a_nX^n$ are nilpotent, so $a_1X + a_2X^2 + \cdots + a_nX^n$ is nilpotent in $R[X]$. Since $a_0 \in U(R)$, $f = a_0 + (a_1X + \cdots + a_nX^n)$ is the sum of a unit and nilpotent element and thus is a unit. It follows that $f \in U(R[X])$. □

We next determine when a polynomial is nilpotent.

Theorem 3.1.2. Let $f \in R[X]$ be given by $f = a_0 + a_1X + \cdots + a_nX^n$. Then f is in the nilradical of $R[X]$ if and only if each $a_i \in Nil(R)$.

Proof. ($\Longrightarrow$) Suppose $f \in Nil(R[X])$. Let P be a prime ideal in R and denote its extension to $R[X]$ by $P[X]$. Consider the map $\pi : R[X] \to R[X]/P[X] = (R/P)[X]$. Since $f \in Nil(R[X])$ we have that f is contained in every prime ideal in $R[X]$, so in particular we have $f \in P[X]$, thus $\pi(f) = \overline{0}$. But this implies $\overline{a_1}, \overline{a_2}, \ldots, \overline{a_n} = \overline{0}$ so each $a_i \in P$. Since P is arbitrary, we get each $a_i \in Nil(R)$.

($\Longleftarrow$) Since $a_0, a_1, \ldots, a_n$ are nilpotent, then $a_0, a_1X, \ldots, a_nX^n$ are nilpotent. Thus their sum $a_0 + a_1X + \cdots + a_nX^n$ is nilpotent. □

Another important ideal in R is the Jacobson radical, denoted $J(R)$. The *Jacobson radical* of a commutative ring R is defined as the intersection of all maximal ideals of R. Note that $J(R) = \{x \in R \mid 1 - xy \text{ is a unit } \forall y \in R\}$ is another characterization of the Jacobson radical. Since every maximal ideal is prime, it follows that we always have $Nil(R) \subseteq J(R)$. In a polynomial ring, however, these two ideals are equal.

Theorem 3.1.3. Let R be a commutative ring and $R[X]$ the polynomial ring. Then $J(R[X]) = Nil(R[X])$.

Proof. From the remarks above we already have $Nil(R[X]) \subseteq J(R[X])$. It remains to show the other containment. Let $f \in J(R[X])$, then in particular, $1-Xf$ is a unit in $R[X]$. If $f = a_0+a_1X+\cdots+a_nX^n$ we have $1-Xf = 1-a_0X-a_1X^2-\cdots-a_nX^{n+1}$. Thus $1 - Xf$ a unit implies that $a_0, \ldots, a_n$ are nilpotent. But then $f \in Nil(R[X])$. So $J(R[X]) \subseteq Nil(R[X])$. It follows that in a polynomial ring the Jacobson radical and nilradical are equal. □

Next we consider another useful characterization, this time for zero divisors. We know that an element $f \in R[X]$ is a *zero divisor* if there exists a nonzero $g \in R[X]$ such that $fg = 0$. The following criterion gives us an alternative way to determine if an element in the polynomial ring is a zero divisor.

Theorem 3.1.4. (McCoy's Theorem) Let $f \in R[X]$, then f is a zero divisor if and only if there exists $c \in R$ with $c \neq 0$ such that $cf = 0$.

Proof. ($\Longrightarrow$) Let $f = a_0 + a_1X + \cdots + a_nX^n$ be a zero divisor in $R[X]$ and suppose $g = b_0 + b_1X + \cdots + b_mX^m$ with $b_m \neq 0$ is a polynomial of least degree in $R[X]$ such that $fg = 0$. Towards a contradiction, suppose $m \geq 1$. If $a_jg = 0$ for every j then

$a_jb_m = 0$ for every j, $j = 0, \ldots, n$. Then $b_mf = 0$, which contradicts the minimality of m since $\deg b_m = 0 < m$. So assume that we do not have $a_jg = 0$ for every j. Let $l = \max\{j : a_jg \neq 0\}$ so $a_lg \neq 0$ but $a_kg = 0$ if $l < k \leq n$. Then we can say:

$$fg = (a_0 + \cdots + a_lX^l)(b_0 + b_1X + \cdots + b_mX^m).$$

But $fg = 0$. So $a_lb_m = 0$. Thus $a_lg \neq 0$, $\deg a_lg < m$, and $fa_lg = a_lfg = a_l \cdot 0 = 0$, a contradiction since we assumed g was a polynomial of least degree equals m satisfying $fg = 0$. This means $m \ngeq 1$, so $m = 0$. Thus, $g = c$ for some $c \in R$.

($\Longleftarrow$) Clear. $\square$

We follow this characterization of zero divisors with a characterization of idempotent elements. An element $e \in R$ is *idempotent* if $e = e^2$. It turns out that an element is idempotent in a polynomial ring only if its constant term is idempotent in the base ring and if all its other coefficients are zero. Thus, the only idempotents in the polynomial ring $R[X]$ are the idempotents from R.

Theorem 3.1.5. Let $f \in R[X]$ be given by $f = a_0 + a_1X + \cdots + a_nX^n \in R[X]$. Then f is idempotent if and only if $a_0 = a_0^2$ and $a_1 = a_2 = \cdots = a_n = 0$.

Proof. ($\Longrightarrow$) Suppose $f = f^2$. Then $a_0 + a_1X + \cdots + a_nX^n = a_0^2 + 2a_0a_1X + \cdots + a_n^2X^{2n}$. Then $a_0 = a_0^2$ as desired. To show $a_1 = a_2 = \cdots = a_n = 0$ we proceed by induction. For the base case note that $a_1 = 2a_0a_1 \implies a_0a_1 = 2a_0^2a_1 \implies a_0a_1 = 2a_0a_1 \implies a_0a_1 = 0$. So $a_1 = 2(a_0a_1) = 2(0) = 0$. Now assume $a_1, \ldots, a_i = 0$ for $2 \leq i < n$. Then $a_{i+1} = a_0a_{i+1} + a_1a_i + a_2a_{i-1} + \cdots + a_ia_1 + a_{i+1}a_0$. Thus $a_{i+1} = 2a_0a_{i+1}$ by assumption. Multiplying both sides by a_0 we have $a_0a_{i+1} = 2a_0^2a_{i+1} = 2a_0a_{i+1}$. Thus $a_0a_{i+1} = 0$. Substituting we have $a_{i+1} = 2(a_0a_{i+1}) = 2(0) = 0$.

($\Longleftarrow$) Clear. $\square$

3.2 Content and Dedekind-Mertens Lemma

Here we discuss briefly the content of a polynomial. We will consider some facts about the content of the product of polynomials over an arbitrary commutative ring. This leads to the well known Dedekind-Mertens Lemma. We begin by defining the content of a polynomial.

Definition 3.2.1. Given $f \in R[X]$, the *content* of f, denoted $c(f)$, is the ideal of R generated by the coefficients of f.

Lemma 3.2.2. For any $f, g \in R[X]$, we have $c(fg) \subseteq c(f)c(g)$.

Proof. Let $f = a_0 + a_1X + \cdots + a_nX^n$ and $g = b_0 + b_1X + \cdots + c_mX^m$. Consider the product fg. Suppose $a \in c(fg)$, then

$$\begin{aligned} a &= \sum_{i=0}^{n+m} r_ic_i && \text{(where } c_i \text{ is the } i^{th} \text{ coefficient of } fg) \\ &= \sum_{i=0}^{n+m} r_i\left(\sum_{j=0}^{i} a_jb_{i-j}\right) \\ &= \sum_{i,j}\left(r_ia_j\right)b_{i-j} \\ &\in c(f)c(g). \end{aligned}$$

$\square$

If R is a GCD domain, that is, a domain where any two elements have a greatest common divisor, we say $f \in R[X]$ is *primitive* if there does not exist a nonunit $d \in R$ with $d \mid a_0, \ldots, a_n$, or equivalently $c(f) \subseteq (d)$. Note that this is also equivalent to $gcd(a_0, \ldots, a_n) = 1$ where the a_i are the coefficients of f. Note that if $c(f) = R$, then f is primitive, but the converse is not true in general. For example, let $f = SX + T \in \mathbb{Q}[S,T][X]$. Then f is primitive over $\mathbb{Q}[S,T]$, but $c(f) = (S,T) \neq \mathbb{Q}[S,T]$.

Note that the statement in Lemma 3.2.2, $c(f)c(g) \supseteq c(fg)$ holds for an arbitrary commutative ring. If we multiply both sides of this by $c(f)^m$ where m is the degree of g we actually get equality. This well known result is called the Dedekind-Mertens Lemma. It can be used to give another proof of Theorem 3.1.4, McCoy's Theorem.

Theorem 3.2.3. (Dedekind-Mertens Lemma) Let $f = a_0 + \cdots + a_n X^n$ and $g = b_0 + \cdots + b_m X^m$ be polynomials in $R[X]$ where R is an arbitrary commutative ring. Then

$$c(f)^m c(f)c(g) = c(f)^m c(fg).$$

Corollary 3.2.4. (McCoy's Theorem) Let $f \in R[X]$ be a zero divisor. Then there exists a nonzero $c \in R$ such that $cf = 0$.

Proof. Suppose that $fg = 0$ where $g \neq 0$. By Dedekind-Mertens Lemma $c(f)^{m+1}c(g) = c(f)^m c(fg) = 0$, where $m = \deg g$. Let t be the smallest positive integer so that $c(f)^t c(g) = 0$ and let $c \in c(f)^{t-1}c(g)$ with c nonzero. Then $c(f)c = 0$, so $cf = 0$. $\square$

3.3 Associate Relations

In Section 2.2 we outlined associate relations in a commutative ring with zero divisors. Given two elements a, b in a commutative R, one might think that if a and b are associated, strongly associated, or very strongly associated as elements of R then they should also be associated in the same way as elements of the polynomial ring. It turns out that this is true for associate and strongly associate, but not for very strongly associate. To distinguish where the associate relations occur, we use $a \sim_{R[X]} b$, $a \approx_{R[X]} b$, and $a \cong_{R[X]} b$ to mean that a and b are associated, strongly associated, and very strongly associated respectively in $R[X]$. As mentioned previously, if $a \sim b$ or $a \approx b$ in R then $a \sim_{R[X]} b$ and $a \approx_{R[X]} b$. The simple proofs are given in the following two theorems.

Theorem 3.3.1. For $a, b \in R$, $a \sim_{R[X]} b$ if and only if $a \sim_R b$.

Proof. ($\Longrightarrow$) Suppose $a \sim_{R[X]} b$, then $aR[X] = bR[X]$. Thus $aR = aR[X] \cap R = bR[X] \cap R = bR$, and so $a \sim_R b$.

($\Longleftarrow$) Since $aR = bR$ we can say that their extensions to the polynomial ring are also equal, that is, $aR[X] = bR[X]$. $\square$

Theorem 3.3.2. For $a, b \in R$, $a \approx_{R[X]} b$ if and only if $a \approx_R b$

Proof. ($\Longrightarrow$) Suppose $a = u(X)b$ for some $u(X) \in U(R[X])$ with $u(X) = c_0 + c_1X + \cdots + c_sX^s$. Then $a = c_0b + c_1bX + \cdots + cbX^s$. So we have $a = c_0b$ where $c_0 \in U(R)$ since $u(X) \in U(R[X])$.

($\Longleftarrow$) Since $a = ub$ for some $u \in U(R)$ if we take $u(X) = u$ to be the constant polynomial we see that $a = u(X)b$ in $R[X]$ where $u(X) \in U(R[X])$. $\square$

Now if $a \cong b$ in R, then we do not necessarily have that $a \cong_{R[X]} b$ in $R[X]$. In [12] Anderson and Valdes-Leon remark that if a nonzero $a \cong_R a$ for every a, then $a \cong_{R[X]} a$ implies $ann(a) \subseteq Nil(R)$. We generalize this remark and provide a proof with the following theorem.

Theorem 3.3.3. For $a, b \in R$, $a \cong_{R[X]} b$ if and only if $a \cong_R b$ and $a = b = 0$ or $ann(b) \subseteq Nil(R)$

Proof. ($\Longrightarrow$) First note that $a \cong_{R[X]} b \implies a \sim_{R[X]} b \implies a \sim_R b$. If $a = b = 0$ we are done. Otherwise assume $a \neq 0$. Let $a = rb$ with $r \in R$. Then in $R[X]$, we also have $a = rb$ where r is a constant polynomial. But then $a \cong_{R[X]} b \implies r \in U(R[X]) \implies r \in U(R)$ since r is a constant. Thus $a \cong_R b$. Now $a \sim_R b$ gives $a = bk$ for some k. Pick $y \in ann(b)$. Then $a = bk + 0X = bk + ybX = b(k + yX)$, thus $k + yX \in U(R[X])$. So $y \in Nil(R)$.

($\Longleftarrow$) Now suppose $a \cong_R b$. Then $a \sim_R b$, so $a \sim_{R[X]} b$. If $a = b = 0$ we are done. Assume $a \neq 0$ and suppose $a = rb$ with $r(X) = r_0 + r_1X + \cdots + r_sX^s$. Then $a = r_0b + r_1bX + \cdots + r_sbX^s$ which can be rewritten as

$$a + 0X + \cdots + 0X^s = r_0b + r_1bX + \cdots + r_sbX^s.$$

So $a = r_0b$ and $r_ib = 0$ for every $i \in \{1, \ldots, s\}$. First $a = r_0b$ implies $r_0 \in U(R)$ since $a \cong_R b$. Next note that $r_ib = 0$ implies $r_i \in ann(b)$ for $i \neq 0$. So each $r_i \in Nil(R)$ by assumption. It follows that $r \in U(R[X])$. $\square$

Definition 3.3.4. A ring R is *présimplifiable* if $x = xy$ implies $x = 0$ or $y \in U(R)$ for any nonunit in R.

In [18], Bouvier explored the properties of présimplifiable rings. In [6] Anderson and Chun looked at when a polynomial ring is présimplifiable. All authors noted the relationship between R présimplifiable and 0 a primary ideal. An ideal $I \subset R$ is *primary* if whenever $xy \in I$, $x \in I$ or $y^n \in I$ for some $n \geq 1$. The following lemma gives an useful characterization of a primary ideal. It is well known but we include the proof here for completeness.

Lemma 3.3.5. An ideal $I \subset R$ is primary if and only if every zero divisor in R/I is nilpotent.

Proof. ($\Longrightarrow$) Let $x + I \in R/I$ be a zero divisor. Then there exists a $y \notin I$ so that $(x+I)(y+I) = xy+I = I$, thus $xy \in I$. Since I is primary, we have $x^n \in I$ for some $n \geq 1$ since $y \notin I$. Then, $(x+I)^n = x^n + I = I$, so $x + I$ is nilpotent.
($\Longleftarrow$) Let $xy \in I$. Suppose $x \notin I$. Then $I = xy + I = (x+I)(y+I)$ with $x + I \neq I$ implies $y + I$ is a zero divisor in R/I. By assumption $y + I$ is nilpotent, so $y^n + I = (y+I)^n = I$ for some $n \geq 1$. But then $y^n \in I$. It follows that I is a primary ideal of R. ☐

In a commutative ring R we always have that $Nil(R) \subseteq Z(R)$. It follows from Lemma 3.3.5 that the ideal (0) is primary if and only if $Nil(R) = Z(R)$. Furthermore, it was shown in [18] that if an ideal $I \subset R$ is primary then R/I is présimplifiable. In the following theorem we give equivalent conditions for a polynomial ring $R[X]$

to be présimplifiable, which includes (0) being a primary ideal in $R[X]$ and R being présimplifiable.

Theorem 3.3.6. Let R be a commutative ring. The following are equivalent:

1. $R[X]$ is présimplifiable,
2. R is présimplifiable and 0 is primary,
3. $a \cong_R a$ and $ann(a) \subseteq Nil(R)$ for all $a \in R$ with $a \neq 0$, and
4. $a \cong_{R[X]} a$ for all $a \in R$ with $a \neq 0$.

Proof. (1) $\implies$ (2) is proven in [18] by Bouvier. We provide another proof here. Suppose $R[X]$ is présimplifiable, then in particular for the constant polynomials in $R[X]$ we have $x = xy \implies x = 0$ or $y \in U(R)$, so R is présimplifiable. Now let $fg = 0$ for $f, g \in R[X]$ with $f = a_0 + a_1X + \cdots + a_nX^n$ and $g = b_0 + b_1X + \cdots + b_mX^m$. Suppose $f \neq 0$. If $g = 0$ then $g \in Nil(R[X])$. If $g \neq 0$, then note that,

$$f = f - fg = f(1 - g)$$

which implies $1 - g \in U(R[X])$ since $R[X]$ is présimplifiable. Then $1 - b_0 \in U(R)$ and each $b_i \in Nil(R)$ for $i \in \{1, \ldots, m\}$. Note that $f = f(1 - g)$ implies $a_0 = a_0 - a_0b_0 \implies a_0b_0 = 0$. So $b_0 \in ann(a_0)$, thus $b_0 \in Nil(R)$ by Thorem 3.3.3. Then each $b_i \in Nil(R)$ for $i \in \{0, \ldots, m\}$ so $g \in Nil(R[X])$. It follows that 0 is primary in $R[X]$, thus 0 is primary in R.

(2) $\implies$ (3) If R is présimplifiable $a \cong_R a$ for all a. Let $x \in ann(a)$, then $xa = 0$. If $a \neq 0$ then $x \in Z(R)$. Since 0 is primary, $Z(R) = Nil(R)$ by Lemma 3.3.5 so $ann(a) \subseteq Nil(R)$.

(3) $\iff$ (4) Follows from Theorem 3.3.3.

(4) $\iff$ (1) Assume $a \cong_{R[X]} a$ for every nonzero $a \in R$. Let $f = fg$ where $f = a_0 + a_1X + \cdots + a_nX^n$ and $g = b_0 + b_1X + \cdots + b_mX^m$. Note that we can write $f = X^k f_1$ where $f_1(0) \neq 0$. Then $X^k f_1 = f = fg = X^k f_1 g$. So $f_1 = f_1 g$, thus we can assume $a_0 \neq 0$. Then $a_0 = a_0 b_0$ and by assumption $b_0 \in U(R)$. We have that $f \neq 0$ and $f = fg \implies f - fg = 0 \implies f(1-g) = 0$. So $1 - g \in Z(R[X])$. Then there exists a $c \in R$ so that $c(1-g) = 0$. In particular, we have

$$\begin{aligned} c(1-g) &= c(1 - b_0 - b_1X - \cdots - b_mX^m) \\ &= c(1-b_0) - cb_1X - \cdots - cb_mX^m \\ &= 0 \end{aligned}$$

so $cb_i = 0$ for every $i \in \{1, \ldots, m\}$. Then $b_i \in ann(c)$ implies $b_i \in Nil(R)$ for every $i \in \{1, \ldots, m\}$ by (3). Since $b_0 \in U(R)$ and each $b_i \in Nil(R)$ for $i \in \{1, \ldots, m\}$, it follows that $g \in U(R[X])$. Thus $f \cong_{R[X]} f$ for every $f \in R[X]$ and it follows that $R[X]$ is présimplifiable. $\square$

Now we briefly consider regular associate relations between two elements of a polynomial ring. It has been established how we define two elements in $R[X]$ being associated, strongly associated, or very strongly associated. Anderson and Chun [6] considered when two elements are regular associates and (strongly) regular associate rings. These definitions are provided below.

Definition 3.3.7. Two elements a and b in a commutative ring R are *regular associates*, denoted $a \sim_r b$, if there exist regular elements $s, t \in R[X]$ with $a = sb$ and

$b = ta$

Definition 3.3.8. A ring R is called *strongly regular associate*, or *sra*, if for all $a, b \in R$, $a \sim b$ implies $a \sim_r b$.

In [6] it was shown that a polynomial ring in n indeterminates is always strongly regular associate. Thus, in a polynomial ring if two elements are associates then they are also regular associates. We include this theorem and proof below for completeness.

Theorem 3.3.9. [6, Theorem 18] For a commutative ring R, $R[X]$ is always strongly regular associate.

Proof. Let $f, g \in R[X]$ and suppose $f \sim g$. Then $f \mid g$ implies $g = fh$ and $g \mid f$ implies $f = gk$ for some $h, k \in R[X]$. Let $c(f), c(g), c(h), c(k)$ be the contents of the polynomials f, g, h, and k respectively. Note that

$$c(g) = c(fh) \subseteq c(f)c(h) \subseteq c(f) \text{ and } c(f) = c(gk) \subseteq c(g)c(k) \subseteq c(g),$$

so we have $c(f) = c(g)$. Thus, we have that $c(f) = c(f)c(h)$. So there must exist some $a \in c(h)$ with $c(f)(1-a) = 0$. This follows from Theorem 76 in [26] which says $IA = I$ where I is finitely generated implies $I(1-a) = 0$ for some $a \in A$. Now, since $1-a$ annihilates the ideal generated by the coefficients of f, we have that $(1-a)f = 0$. Consider $h_1 = h + (1-a)X^{n+1}$ where $n = \deg h$. Then $c(h_1) = c(h) + R(1-a) = R$. Thus h is regular. It follows that,

$$fh_1 = f(h + (1-a)X^{n+1}) = fh + (1-a)fX^{n+1} = fh = g.$$

So there exists a $h_1 \in R[X]$ that is regular so that $g = fh_1$. Similarly, there exists a $k_1 \in R[X]$ regular so that $f = gk_1$. It follows that $R[X]$ is strongly regular associate. $\square$

Corollary 3.3.10. Let $f, g \in R[X]$ with R commutative. Then $f \sim g$ in $R[X]$ if and only if $f \sim_r g$ in $R[X]$.

Proof. By Theorem 3.3.9, $R[X]$ is always strongly regular associate, thus $f \sim g \Rightarrow f \sim_r g$ in $R[X]$. Conversely, $f \sim_r g$ always implies $f \sim g$. $\square$

3.4 Irreducibles and Atomic Factorization

The definitions of irreducible, strongly irreducible, very strongly irreducible, and m-irreducible discussed in Section 2.1 remain the same in a polynomial ring but the implications vary slightly. For a nonzero element in a commutative ring R we have the following implications.

$$\text{very strongly irreducible} \Longrightarrow m\text{-irreducible} \Longrightarrow \text{strongly irreducible} \Longrightarrow \text{irreducible}$$

Note that 0 is never m-irreducible in $R[X]$, for that would imply $R[X]/(0) = R[X]$ is a field, which is impossible. Also note that in a polynomial ring m-irreducible and very strongly irreducible coincide. We need the following fact.

Lemma 3.4.1. [24, Corollary 6.3] Let R be a commutative ring and let $a \in R$. Then (a) is idempotent if $(a) = (e)$ where $e = e^2$ for some $e \in R$.

Lemma 3.4.2. [5, Theorem 2.9] Let R be a commutative ring and let $a \in R$. Then a is m-irreducible if and only if (a) is an idempotent maximal ideal of R or $a \neq 0$ is very strongly irreducible.

Proof. ($\Longrightarrow$) Let a be m-irreducible. If $a = 0$ then (a) is an idempotent maximal ideal of R and we are done. So assume $a \neq 0$. If $(a) = (a)^2$ then (a) is an idempotent ideal, so by Lemma 3.4.1, $(a) = (e)$ where $e \in R$ is idempotent. Note that for any $f \in R$, $(e, f) = (e, (1-e)f) = (e + (1-e)f)$. Since (a) is a maximal principal ideal of R, then $(e, f) = (e)$ or $(e, f) = R$. Hence, $R/(e)$ has no proper principal ideals and hence is a field. So (a) is an idempotent maximal ideal.

Now suppose $(a) \neq (a)^2$ and let $a = bc$. Then $(a) \subseteq (b)$ and $(a) \subseteq (c)$. Since (a) is maximal in the set of principal ideals, this implies $(a) = (b)$ or $(b) = R$ and $(a) = (c)$ or $(c) = R$. We cannot have both $(a) = (b)$ and $(a) = (c)$ since this implies $(a) = (b)(c) = (a)(a) = (a)^2$, a contradiction. So either $(a) = (b)$ and $(c) = R$ or $(a) = (c)$ and $(b) = R$. Thus $a = bc$ implies b or c is a unit and so a is very strongly irreducible.

($\Longleftarrow$) If (a) is a maximal ideal, then a is m-irreducible. Otherwise if a is very strongly irreducible then this implies that a is m-irreducible. $\square$

Theorem 3.4.3. Let $f \in R[X]$ where $f \neq 0$. Then f is m-irreducible if and only if f is very strongly irreducible.

Proof. ($\Longrightarrow$) Suppose $f \in R[X]$ is m-irreducible. By Lemma 3.4.1 f is m-irreducible if and only if the ideal generated by f, (f), is an idempotent maximal ideal or $f \neq 0$ is very strong irreducible. Suppose (f) is a maximal idempotent ideal. Then $(f) = (g)$

where g is idempotent in $R[X]$. By Theorem 3.1.5, $g \in R$ so $(f) = (g)$ where $g \in R$ is idempotent. Then $(f) = (g) \subsetneq (g, X) \subsetneq R[X]$, which contradicts the maximality of (f). Thus (f) not an idempotent maximal ideal implies f is very strongly irreducible.

$(\Longleftarrow)$ Clear. $\square$

Thus we have the following implications for all nonzero $f \in R[X]$:

$$\begin{tikzcd} & & & \text{prime} \arrow[d, Rightarrow] \\ \text{very strongly irreducible} \arrow[r, Leftrightarrow] & m\text{-irreducible} \arrow[r, Rightarrow] & \text{strongly irreducible} \arrow[r, Rightarrow] & \text{irreducible.} \end{tikzcd}$$

Note that above, very strongly irreducible if and only if m-irreducible only holds for nonzero f. Also, aside from very strong irreducible if and only if m-irreducible, none of the implications in the above diagram can be reversed. Specific examples of this are given at the end of this section. Now note that each type of irreducibility leads to a different form of atomicity. In a polynomial ring $R[X]$ we have the following implications.

$$\begin{tikzcd} & & p\text{-atomic} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{ACCP} \arrow[d, Rightarrow] \\ \text{very strongly atomic} \arrow[r, Leftrightarrow] & m\text{-atomic} \arrow[r, Rightarrow] & \text{strongly atomic} \arrow[r, Rightarrow] & \text{atomic} \end{tikzcd}$$

Again, aside from very strongly atomic if and only if m-atomic, none of the above implications can be reversed. In the following section we provide several examples to show the distinctions between different types of irreducible conditions and atomic conditions shown in the diagrams above.

3.5 Examples

In this section we give a number of examples of elements and rings that satisfy certain factorization properties discussed in the previous sections.

Example 3.5.1. An element that is strongly irreducible and prime but not very strongly irreducible.

Let $\bar{p}$ be in $\mathbb{Z}_m \subset \mathbb{Z}_m[X]$ where $p \mid m$ with p prime in $\mathbb{Z}$. Note that $\bar{p}$ is prime in $\mathbb{Z}_m$ and in $\mathbb{Z}_m[X]$. Also note that for any $\bar{a} \in \mathbb{Z}_m$ we have that $\bar{a}$ is a unit if and only if $(a, m) = 1$ in $\mathbb{Z}$. Then $\bar{p}$ is strongly irreducible. To see this, since $p \mid m$ we can write $m = p^k q_1^{n_1} \cdots q_s^{n_s}$ with the q_i prime. Then we have $\mathbb{Z}_m \cong \mathbb{Z}_{p^k} \times \mathbb{Z}_{q^{n_1}} \times \cdots \times \mathbb{Z}_{q^{n_s}}$ and $\bar{p} \mapsto (\bar{p}, 1, \ldots, 1)$. Since $\bar{p}$ is prime, it is also irreducible, so $(\bar{p}, 1, \ldots, 1)$ irreducible implies that $\bar{p}$ is irreducible in $\mathbb{Z}_{p^k}$. Since $\mathbb{Z}_{p^k}$ is local it is présimplifiable, so all types of irreducible elements coincide, thus $\bar{p}$ irreducible implies strongly irreducible. Since an element is strongly irreducible in a direct product if it is strongly irreducible in one coordinate and a unit in all others, it follows that if $p \mid m$, then $\bar{p}$ is strongly irreducible in $\mathbb{Z}_m$. A similar argument shows that $\bar{p}$ is m-irreducible in $\mathbb{Z}_m$. Since $\bar{0}$ is primary in $\mathbb{Z}_m$, $\mathbb{Z}_m[X]$ is présimplifiable by Theorem 3.3.6, so $\bar{p}$ is also strongly irreducible in $\mathbb{Z}_m[X]$. However, $\bar{p}$ is very strongly irreducible in $\mathbb{Z}_m$ and $\mathbb{Z}_m[X]$ if and only if $p^2 \mid m$ or $p = m$ and $\bar{p}$ is m-irreducible in $\mathbb{Z}_m[X]$ if and only if $p^2 \mid m$.

If $\bar{p}$ is very strongly irreducible in $\mathbb{Z}_m$ then $(\bar{p}, 1, \ldots, 1)$ very strongly irreducible implies that $\bar{p}$ is very strongly irreducible in $\mathbb{Z}_{p^k}$ and is nonzero unless $\mathbb{Z}_{p^k}$ is an integral domain. Thus if $\bar{p} = \bar{0}$, $\mathbb{Z}_{p^k}$ is an integral domain which implies $k = 1$, so $p = m$. Otherwise, if $p \neq m$ and $\bar{p}$ is nonzero in $\mathbb{Z}_{p^k}$ we must have $p^l \mid m$ for some

$l \geq 2$, so $p^2 \mid m$. Conversely, If $p = m$ in $\mathbb{Z}$, then $\mathbb{Z}_m$ is a field, thus $\bar{p} = \bar{0}$ is very strongly irreducible. If $p^2 \mid m$, then $\bar{p} \neq \bar{0}$. So $(\bar{p}, 1, \ldots, 1)$ is a unit in every coordinate except for one and $\bar{p} \in \mathbb{Z}_{p^k}$ is very strongly irreducible. Thus $\bar{p}$ is very strongly irreducible in $\mathbb{Z}_m$ and in $\mathbb{Z}_m[X]$. By Theorem 4.4.3 this also gives that $\bar{p}$ is m-irreducible in $\mathbb{Z}_m[X]$ if and only if $p^2 \mid m$.

Example 3.5.2. [12, Example 2.3] An element that is irreducible but not strongly irreducible.

Let $R = F[X, Y, Z]/(X - XYZ)$ where F is a field, and x, y, z be the images of X, Y, Z in R respectively. Then x is irreducible but is not strongly irreducible. To see that x is irreducible note that $X \in F[X, Y, Z]$ is prime since $F[X, Y, Z]/(X) \cong F[Y, X]$ is an integral domain.

Note that $x = xyz$. Suppose x is strongly irreducible, then $x \approx z$ or $x \approx xy$. If $x \approx z$ then $z = ux$ for some $u \in R$. Then $x = x^2yu \implies (x) = (x^2)$. Then we have $(X) + X(1 - YZ) = (X^2) + X(1 - YZ) \implies (1) + (1 - YZ) = (X) + (1 - YZ)$. So $F[X, Y, Z] = (X) + (1 - YZ) \implies F[X, Y, Z] = (X, 1 - YZ)$ a contradiction, so $x \not\approx z$.

Now suppose $x \approx xy$. Then $xy = x\bar{f}$ for some $\bar{f} \in U(R)$ with $\bar{f}$ the image of some $f \in F[X, Y, Z]$. Then $x\bar{f} - xy = \bar{0} \implies fX - XY \in (X - XYZ) \implies X(f - Y) \in X(1 - YZ)$. So $f - Y \in (1 - YZ) \implies f = Y + h(1 - YZ)$. If f is a unit then $(X, (1 - YZ)) = R$, a contradiction, so $x \not\approx xy$.

Example 3.5.3. An element that is irreducible but not prime.

Let $X \in R[X]$ where R is any indecomposable ring that is not an integral domain.

Then X is irreducible by Theorem 4.1.2, but X is not prime since $R[X]/(X) \cong R$ is not a domain. For example, take $R = \mathbb{Z}_4$, then R is indecomposable but not a domain, so X is irreducible in $R[X]$ but not prime.

Example 3.5.4. A ring that satisfies ACCP but is not p-atomic.

$\mathbb{Z}_4[X]$ is Noetherian since $\mathbb{Z}_4$ is, so it satisfies ACCP, but it is not p-atomic by Theorem 4.2.6. In particular, not every regular element is the product of primes since X is regular but not prime.

Example 3.5.5. [12, Example 2.3] A ring that satisfies ACCP but is not strongly atomic.

Let $R = F[X,Y,Z]/(X - XYZ)$ where F is a field, and x, y, z be the images of X, Y, Z in R respectively. Then R is Noetherian, and thus satisfies ACCP, but it is not strongly atomic because x is not a product of strongly irreducible elements. For if $x = x_1 \cdots x_n$ where each x_i is strongly irreducible, then x irreducible gives $x \sim x_i$ for some i and hence x is strongly irreducible, a contradiction (Example 3.5.2).

Example 3.5.6. [25] A ring that satisfies ACCP but the polynomial ring does not.

Let k be a field, $A_1, A_2, \ldots,$ indeterminates over k and $S = k[A_1, A_2, \ldots]/(\{A_n(A_{n-1} - A_n) : n \geq 2\})k[A_1, A_2, \ldots]$. Let a_n be the image of A_n in S. Then let R be the localization of S at the ideal $(a_1, a_2, \ldots)S$. Then R satisfies ACCP but $R[X]$ does not satisfy ACCP.

Example 3.5.7. [19] A ring with no atoms such that the polynomial ring is very strongly atomic.

Let $\mathbb{F}$ be a perfect field of characteristic p and $\{x_1, x_2, \ldots, x_n, \ldots,\}$ be a countable collection of indeterminates. Let $T := \mathbb{F}\big[\{x_1^{\alpha_1}, x_2^{\alpha_2}, \ldots\} \mid \alpha_i \in \mathbb{Q}^+ \bigcup\{0\}\big]$. Let $I :=< \{\prod_{i=1}^{\infty} x_i^{\beta_i}\} >$ where $\beta_i = 0$ for all but finitely many i and $\sum_{i=1}^{\infty} \beta_i > 1$. Let $R := T/I$, then R is a 0-dimensional quasi-local ring that is antimatter (has no atoms) but the polynomial ring is very strongly atomic.

Now we discuss a useful construction for finding examples of commutative rings with zero divisors with certain factorization properties called the idealization of a commutative ring R with a unitary R-module N, denoted $R(+)N$.

Definition 3.5.8. Given a commutative ring R and a unitary R-module N, the *idealization* or *trivial extension* of R and N is the ring $R(+)N = \{(r, n) \mid r \in R, n \in N\}$ where addition is given by $(r, m) + (s, n) = (r + s, m + n)$ and multiplication is given by $(r, m)(s, n) = (rs, rn + sm)$.

Suppose R is an integral domain and N an R-module, and consider $R(+)N$. Note that $(0, n)(0, m) = (0, 0 \cdot m + 0 \cdot n) = (0, \bar{0})$ so $0 + N$ is an ideal with nilpotent index 2. The set of zero divisors is given by $Z(R(+)N) = \{(r, n) \mid r \in Z(N)\}$ and it is also easily seen that $U(R(+)N) = \{(r, n \mid r \in U(R)\}$. The following two theorems characterize irreducibles in $R(+)N$ and give important factorization properties about the idealization $R(+)M$. The proofs can be found in [12].

Theorem 3.5.9. [12, Proposition 5.1] Let R be an integral domain, N an R-module, and $R_1 = R(+)N$.

1. If $0 \neq a \in R$ is irreducible, (a, m) is very strongly irreducible.

2. For $0 \neq a \in R$, the following are equivalent:

   (a) a is irreducible

   (b) $(a, 0)$ is irreducible

   (c) $(a, 0)$ is strongly irreducible

3. For $0 \neq n \in N$,

   (a) $(0, n)$ is irreducible if and only if $n = am$ implies $Rn = Rm$ if and only if Rn is a maximal cyclic submodule of N.

   (b) $(0, n)$ is strongly irreducible if and only if $n = am$ implies $n = um$ where $u \in U(R)$, and

   (c) $(0, n)$ is very strongly irreducible if and only if $n = am$ implies $a \in U(R)$.

Theorem 3.5.10. [12, Proposition 5.2] Let R be an integral domain, N an R-module, and $R_1 = R(+)N$.

1. If R satisfies ACCP, then every ascending chain of principal ideals of R_1 containing a principal ideal of the form $R_1(a, n)$ where $a \neq 0$ stops.

2. R_1 satisfies ACCP if and only if R satisfies ACCP and N satisfies ACCC (ascending chain condition on cyclic submodules)

3. R_1 is a BFR if and only if R is a BFD and N is a BF-module, i.e., for $0 \neq n \in N$, there exists a natural number $N(n)$ so that $n = a_1 \cdots a_{s-1} n_s$ implies $s \leq N(n)$.

4. R_1 is atomic if R satisfies ACCP and N satisfies MCC (every cylic submodule of N is contained in a maximal (not necessarily proper) cyclic submodule).

5. R_1 is présimplifiable if and only if N is présimplifiable, i.e. $n = an$ implies

$n = 0$ or $a \in U(R)$.

We can use Theorem 3.5.9 and Theorem 3.5.10 to give some examples of rings with zero divisors with certain factorization properties. The following example gives a straightforward example of a ring R that is atomic but does not satisfy ACCP.

Example 3.5.11. [12, Example 5.3] A ring that is atomic but does not satisfy ACCP.

Let $R = \mathbb{Z}_{(2)}$, $N = \mathbb{Z}_2 \oplus \mathbb{Z}_{2^\infty}$, and $R_1 = R(+)N$. Here $\mathbb{Z}_{(2)} = \{\frac{a}{b} \mid 2 \nmid b\}$ is the localization of $\mathbb{Z}$ at the prime ideal 2, $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, and $\mathbb{Z}_{2^\infty} = \bigcup_{n=1}^{\infty} \mathbb{Z}_{2^n} = \{\frac{a}{2^n} + \mathbb{Z} \mid 0 \leq a \leq 2^n - 1\}$, which is the 2-primary component of $\mathbb{Q}/\mathbb{Z}$.

<u>R_1 is atomic</u>: By Theorem 3.5.10, R_1 is atomic if R satisfies ACCP and N satisfies MCC. Note that $\mathbb{Z}$ is Noetherian, so $\mathbb{Z}_{(2)}$ is Noetherian, thus $\mathbb{Z}_{(2)}$ satisfies ACCP. Now, consider a cyclic submodule of N generated by $(\bar{1}, a)$ where a is any element in $\mathbb{Z}_{2^\infty}$. Suppose $\mathbb{Z}_{(2)}(\bar{1}, a)$ is contained in another cyclic submodule, say $\mathbb{Z}_{(2)}(c, d)$. So we have $\mathbb{Z}_{(2)}(\bar{1}, a) \subseteq \mathbb{Z}_{(2)}(c, d)$. This implies that $c = \bar{1}$, and that we can write $(\bar{1}, a) = r(\bar{1}, d)$ for some $r \in \mathbb{Z}_{(2)}$. Then $r \cdot \bar{1} = \bar{1}$ implies that $r \in U(\mathbb{Z}_{(2)})$, so $(\bar{1}, a)$ and (c, d) differing by a unit multiple means $\mathbb{Z}_{(2)}(\bar{1}, a) = \mathbb{Z}_{(2)}(c, d)$. So $\mathbb{Z}_{(2)}(\bar{1}, a)$ is a maximal cyclic submodule.

Now consider $\mathbb{Z}_{(2)}(\bar{0}, b)$ where b is any element in $\mathbb{Z}_{2^\infty}$. Then $\mathbb{Z}_{(2)}(\bar{0}, b) \subseteq \mathbb{Z}_{(2)}(\bar{1}, \frac{b}{2})$ since $(\bar{0}, b) = 2(\bar{1}, \frac{b}{2})$. So every cyclic submodule of N is maximal or contained in a maximal cyclic submodule, thus N satisfies MCC. It follows that R_1 is atomic.

<u>R_1 does not satisfy ACCP</u>: By Theorem 3.5.10, R_1 satisfies ACCP if R satisfies ACCP and N satisfies the ascending chain condition on cyclic submodules (ACCC).

By previous remarks we know that R has ACCP. Consider

$$\mathbb{Z}_{(2)}\big(\bar{0}, \frac{1}{2} + \mathbb{Z}\big) \subset \mathbb{Z}_{(2)}\big(\bar{0}, \frac{1}{2^2} + \mathbb{Z}\big) \subset \mathbb{Z}_{(2)}\big(\bar{0}, \frac{1}{2^3} + \mathbb{Z}\big) \subset \cdots .$$

This is an infinite ascending chain of cyclic submodules of N, so N does not satisfy ACCC, thus R_1 does not satisfy ACCP.

Next we consider a particular construction using the idealization of a module, $R = R(+)R/M$ where M is a maximal ideal and R is a quasi-local integral domain. We give some of its factorization properties and then consider the polynomial ring over R in one indeterminate in the following two theorems.

Theorem 3.5.12. Let (D, M) be a quasi-local domain with maximal ideal M and let $R = D(+)D/M$, then we have the following:

1. R is always atomic,
2. R satisfies ACCP if and only if D satisfies ACCP, and
3. R is a BFR if and only if D is a BFD.

Proof. (1) Let (D, M) be a quasi-local domain with maximal ideal M, then $D(+)D/M$ is atomic. To see this first note that $U(D(+)D/M) = U(D)(+)D/M = \{(r, \bar{n}) \mid r \notin M\}$. So any nonzero nonunit is of the form $(r, \bar{n})$ where $r \in M$ and $\bar{n} = n + M$ is in the D-module D/M. Suppose $r = 0$ and $\bar{n} \neq \bar{0}$. Then if $(0, \bar{n}) = (a, \bar{x})(b, \bar{y})$ then $ab = 0$ gives $a = 0$, $b = 0$, or $a = b = 0$ since D is a domain. If both a and b are zero then $(0, \bar{n}) = (0, \bar{0})$ a contradiction, so only one is 0, say a is. Then $(0, \bar{n}) = (0, \bar{x})(b, \bar{y}) = (0, b\bar{y})$. Now note that since $\bar{n} = b\bar{y}$ we have,

$$n + M = b(y + M) \implies n + M = by + M \implies n - by \in M$$

but $n+M$ nonzero implies $n \notin M$ so $by \notin M$, so $b \notin M$, which implies $b \in U(D)$. So $(b, \bar{y})$ is a unit. Similarly, if $b = 0$ then $(a, \bar{x})$ is a unit. It follows that $(0, \bar{n})$ is an atom.

If $r \neq 0$, then suppose $(r, \bar{n}) = (a, \bar{x})(b, \bar{y}) = (ab, (ay + bx) + M)$. So $r = ab$ implies $a \in M$ or $b \in M$. If $a \in M$ but $b \notin M$ then $(b, \bar{y}) \in U(D(+)D/M)$ and this implies $(r, \bar{n})$ is an atom. Similarly, $(r, \bar{n})$ is an atom if $a \notin M$ but $b \in M$.

If both $a, b \in M$ then $ay, bx \in M$ so,

$$(r, \bar{n}) = (r, \bar{0}) = (a, \bar{0})(b, \bar{0}) = (ab, \bar{0}) = (a, \bar{1})(b, \bar{1}).$$

But $(s, \bar{1})$ is an atom for any $0 \neq s \in D$ with $s \in M$. To see this note that if $(s, \bar{1}) = (c, \bar{k})(d, \bar{l})$ with both $c, d \in M$ then $(s, \bar{1}) = (cd, \bar{0})$ a contradiction. So either c or d is in $R - M$, which implies either $(c, \bar{k})$ or $(d, \bar{l})$ is a unit. It follows that R is atomic, in fact every nonzero nonunit is either an atom or the product of two atoms.

(2) Suppose $(D/M)\bar{n_0} \subsetneq (D/M)\bar{n_1} \subsetneq \cdots$ is an ascending chain of cyclic submodules in D/M. Since D/M is a simple module, the only cyclic submodules are $(D/M)\bar{0}$ and $(D/M)\bar{n} = D/M$ where $\bar{n}$ is any nonzero element in D/M. So any chain has at most length two before it stops, thus D/M satisfies ACCC. It follows from Theorem 3.5.10 that R satisfies ACCP if and only if D satisfies ACCP.

(3) Certainly D/M is a BF-module. Thus by Theorem 3.5.10 $D(+)D/M$ is a BFR if and only if D is a BFD. $\square$

Theorem 3.5.13. Let (D, M) be a quasi-local domain with maximal ideal M and let $R = D(+)D/M$, then we have the following:

1. $R[X]$ satisfies ACCP if and only if R satisfies ACCP, and

2. $R[X]$ is a BFR if and only if R is a BFR.

Proof. (2) ($\Longrightarrow$) Clear.

($\Longleftarrow$) In [25] Heinzer and Lantz show that ACCP extends to the polynomial ring if R is a zero-dimensional ring or if R is quasi-local with finitely many associated primes. Frohn refines this result in [23] by showing that ACCP extends to the polynomial ring if R has only finitely many associated primes, thus the quasi-local condition is unnecessary. Note that R has only two associated primes, $0(+)D/M$ and $M(+)D/M$ where $0(+)M$ annihilates every nonunit in R and $M(+)D/M$ annihilates elements of the form $(0, \bar{n})$. Thus if R satisfies ACCP we have $R[X]$ satisfies ACCP.

(3) ($\Longrightarrow$) Clear.

($\Longleftarrow$) In [8, Theorem 12] Anderson and Ganatra show that the bounded factorization property extends to the polynomial ring if zero is primary in R and if $\bar{J} = 0$ where $J = nil(R)$ and $\bar{J} = \bigcap_{i=1}^{\infty} J^i$. Since $0(+)D/M$ is prime in R it is primary. Also note that $nil(R) = nil(D)(+)D/M = 0(+)D/M$. Note that $(0, \bar{n})^k = (0, \bar{0})$ for any $k \geq 2$. Then $\bigcap_{i=1}^{\infty}(0(+)D/M)^i = (0, \bar{0})$. Thus by [8, Thoerem 13] we have that if R is a BFR then $R[X]$ is a BFR. $\square$

CHAPTER 4
UNIQUE FACTORIZATION IN $R[X]$

Our goal in this chapter is to characterize when a polynomial ring is a unique factorization ring. Unique factorization rings have been studied by Anderson and Markanda in [9] and [10], by Fletcher in [20], by Bouvier in [18], by Anderson and Valdes-Leon in [12] and [14], by Anderson, Chun, and Valdes-Leon in [7], and by Agargün, Anderson, and Valdes-Leon in [1]. Here we will refine and extend those results to polynomial rings with zero divisors. In particular we will characterize when a polynomial ring is a unique factorization ring in the sense of Bouvier and Galovich, and also when it is a unique factorization ring in the sense of Fletcher. We also consider when a polynomial ring is a reduced unique factorization ring and when it is a weak unique factorization ring. A characterization of when a polynomial ring is factorial is also given.

4.1 Factoring Powers of Indeterminates

To begin, given a commutative ring R, we focus on properties of the indeterminate X in the polynomial ring $R[X]$. We are motivated by the question: when does X or powers of X have unique factorization? We start by noting when X is prime or irreducible. The following theorem is well known.

Theorem 4.1.1. X is prime in $R[X]$ if and only if R is an integral domain.

Proof. ($\Longrightarrow$) Let X be prime in $R[X]$, then (X) is a prime ideal. Thus $R[X]/(X) \cong R$ is an integral domain.

($\Longleftarrow$) Note that $R[X]/(X) \cong R$ where R is an integral domain. Thus (X) is a prime ideal, so X is prime. $\square$

Now we consider when X is irreducible in $R[X]$. In [12, Theorem 6.4], Anderson and Valdes-Leon give conditions for X to be the finite product of irreducibles. They conclude that X is irreducible if and only if R is an indecomposable ring. We provide a direct proof here.

Theorem 4.1.2. [12, Theorem 6.4] Let R be a commutative ring. Then X is irreducible in $R[X]$ if and only if R is indecomposable.

Proof. ($\Longrightarrow$) Let X be irreducible and suppose $R \cong R_1 \times R_2$ with R_1 and R_2 nonzero. Then $R[X] \cong R_1[X] \times R_2[X]$ and $X \mapsto (X, X)$ where $(X, X) = (X, 1) \cdot (1, X)$. But $(X, X) \not\sim (X, 1)$ and $(X, X) \not\sim (1, X)$, so (X, X) is not irreducible, a contradiction. Thus R is indecomposable.

($\Longleftarrow$) Conversely, suppose that R is indecomposable, so $R[X]$ is indecomposable. Let $X = fg$ with $f, g \in R[X]$ where $f = a_0 + a_1X + \cdots + a_nX^n$ and $g = b_0 + b_1X + \cdots + b_mX^m$. Note that $a_0b_0 = 0$ and $a_0b_1 + a_1b_0 = 1$. If a_0 and b_0 are both zero, then $1 = a_0b_1 + a_1b_0 = 0 \cdot b_1 + a_1 \cdot 0 = 0$, a contradiction. So we cannot have both a_0 and b_0 equal to zero. Assume $a_0 \neq 0$. Then $a_0 = a_0(a_0b_1 + a_1b_0) = a_0^2b_1 + a_0a_1b_0 = a_0^2b_1$. So $(a_0)^2 = (a_0)$, which implies $(a_0) = (e)$ for some idempotent $e \neq 0$. Since R is indecomposable, the only idempotents are 0 and 1, so we must have $e = 1$. So

$(a_0) = (1)$ implies that a_0 is a unit. Then $a_0 b_0 = 0$ implies $b_0 = 0$. Then $X \mid g$ since g has a zero constant term in $R[X]$ and $g \mid X$ since we assumed that $X = fg$. It follows that $X \sim g$. □

In [12] Anderson and Valdes-Leon showed that X is a product of n irreducible elements in $R[X]$ if and only if R is a finite direct product of n indecomposable rings. We provide a modified proof of the result in [12] below.

Theorem 4.1.3. [12, Theorem 6.4] Let R be a commutative ring. Then X is a product of n atoms if and only if R is a direct product of n indecomposable rings.

Proof. Let X be a product of n atoms. Suppose $R \cong R_1 \times \cdots \times R_m$, with the R_i not necessarily indecomposable. Then $R[X] \cong R_1[X] \times \cdots \times R_m[X]$ and $X \mapsto (X, \ldots, X)$. Since an atom in the direct product is irreducible in one coordinate and a unit in all the others, we cannot write X as a product of fewer than m atoms, thus $m \leq n$. So we can write $R = R_1 \times \cdots \times R_m$ where each R_i is indecomposable. Suppose $X = f_1 \cdots f_n$ where $f_i = (f_{i_1}, \ldots, f_{i_m})$ so that in $R_i[X]$ we have $X = f_{1_i} \cdots f_{n_i}$. Since X is irreducible, exactly one f_{j_i} is not a unit. If $n > m$ then for some j, $f_{j_1}, \ldots, f_{j_m}$ are all units, a contradiction. Thus $m = n$.

Conversely, if R is the direct product of n indecomposable rings, say $R \cong R_1 \times \cdots \times R_n$ we have $R[X] \cong R_1[X] \times \cdots \times R_n[X]$. Then $X \mapsto (X, \ldots, X) = X_1 \cdots X_n$ where each X_i has X in the i^{th} coordinate and 1 in all the others. Since each R_i is indecomposable, X is irreducible in each $R_i[X]$ by Theorem 4.1.6. Since atoms in a direct product are irreducible in one coordinate and a unit in all others, each X_i is irreducible, so X is the product of n atoms. $\square$

Corollary 4.1.4. When X is a finite product of atoms, the factorization is unique up to order and associates.

Proof. Suppose that X is a finite product of atoms. Then by Theorem 4.1.8 R is a finite product of indecomposable rings and we have $X = X_1 \cdots X_n$ where each $X_i = (1, \ldots, 1, X, 1, \ldots, 1)$ with X in the i^{th} coordinate is irreducible. Suppose that there is another factorization of X into atoms, then it must have length n again by Theorem 4.1.8. So let $X = f_1 \cdots f_n$ with the f_i irreducible. By the characterization of irreducibles in the direct product, each f_i has exactly one coordinate that is not a unit. We can reorder so that each f_i has the i^{th} coordinate a nonunit. Then $f_i = (f_{i_1}, \ldots, f_{i_n})$ and all the f_{i_j} are units except $f_{1_1}, \ldots, f_{n_n}$. So in $R_i[X]$, $X = f_{1_i} \cdots f_{n_i}$, so all the f_{i_j} are units except f_{i_i}. If we let u be the product of all the f_{i_j} with $j \neq i$, then we have $X = u \cdot f_{i_i}$. So $f_{i_i} = u^{-1}X$. So each $f_i = uX_i$. It follows that the factorization of X is unique up to order and associates. $\square$

While the factorization of X is unique, factorization of X^n for $n > 1$ does not have to be. As an example, consider the factorization of X^n in $\mathbb{Z}_4[X]$. We have that $X^2 = X \cdot X = (X + \bar{2})^2$ are two factorizations of X^2 into irreducibles. Note that $X^5 = X \cdot X \cdot X \cdot X \cdot X = X(X^2 + \bar{2})^2$, so any two factorizations of X^n for $n > 1$ need not have the same length. Here we used that X and $X^n + \bar{2}$ are irreducible. X is irreducible since $\mathbb{Z}_4$ is an indecomposable ring. To see that $X^n + \bar{2}$ is irreducible consider the ring homomorphism $\phi : \mathbb{Z}_4[X] \to \mathbb{Z}_2[X]$ where $X \mapsto X$, $\bar{1}$ and $\bar{3}$ map to

$\bar{1}$, and $\bar{0}$ and $\bar{2}$ map to $\bar{0}$. Then $X^n + \bar{2} \mapsto X^n$ so nonunit factors of $X^n + \bar{2}$ must map to a power of X in $\mathbb{Z}_2[X]$ and thus the constant term of each nonunit factor must be divisible by 2.

If $X^n + \bar{2} = fg$ where $f, g \in \mathbb{Z}_4[X]$ such that $f = a_0 + \cdots + a_s X^s$ and $g = b_0 + \cdots + b_t X^t$ then $a_0 b_0 = \bar{2}$. So one of a_0 or b_0 must be $\bar{2}$ and the other is $\bar{1}$ or $\bar{3}$. Say $a_0 = \bar{2}$, then $b_0 = \bar{1}$ or $\bar{3}$. Then the image of g in $\mathbb{Z}_2[X]$ is $\bar{1}$ or $\bar{3}$, so each b_i for $i > 0$ is either $\bar{0}$ or $\bar{2}$ and hence nilpotent. So g is a unit in $\mathbb{Z}_4[X]$ which implies $X^n + \bar{2}$ is strongly associated to f. We can generalize this argument to say that $X^n + \bar{p}$ is irreducible in $\mathbb{Z}_{p^m}[X]$ for any prime $p \in \mathbb{Z}$ by using the map $\phi : \mathbb{Z}_{p^m}[X] \to \mathbb{Z}_p[X]$.

In the example above the factorization of X^n is not unique but we can still say something about the length of possible factorizations of X^n. Let $L(X^n)$ and $l(X^n)$ represent the longest and shortest lengths of a factorization of X^n into irreducibles in $\mathbb{Z}_4[X]$ respectively, and let $\rho(X^n) = L(X^n)/l(X^n)$. Then we have the following theorem.

Theorem 4.1.5. In $\mathbb{Z}_4[X]$ we have, $L(X^n) = l(X^n)$ if $n = 1$ and and for $n > 1$

$$L(X^n) = n, \;\; l(X^n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}, \text{ and } \rho(X^n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ n/3 & \text{if } n \text{ is odd} \end{cases}.$$

Proof. Let $n = 1$. Since $\mathbb{Z}_4$ is indecomposable, X is irreducible in $\mathbb{Z}_4[X]$ by Theorem 4.1.3, so we have $L(X^n) = l(X^n) = 1$. Also in this case $\rho(X^n) = 1$.

Now suppose $n > 1$. Note that we can factor $X^n = \underbrace{X \cdots X}_{n \text{ times}}$, so $L(X^n) \geq n$. If $L(X^n) > n$ then $X^n = f_1(X) \cdots f_m(X)$ where $m \geq n+1$ and each $f_i(X)$ is irreducible.

Consider the surjective homomorphism $\phi : \mathbb{Z}_4[X] \to \mathbb{Z}_2[X]$ where $X \mapsto X$ and $a \mapsto \bar{1}$ if $a \equiv 1 \mod 2$ and $a \mapsto \bar{0}$ if $a \equiv 0 \mod 2$ for $a \in \mathbb{Z}_4$. Since $\mathbb{Z}_2[X]$ is a unique factorization domain the image of X^n is a product of n irreducibles. Then $X^n = f_1(X) \cdots f_m(X) \mod 2$ implies that at least one of the f_j must map to $\bar{1}$. Such an f_j must be of the form $a_0 + a_1X + \cdots + a_sX^s$ where a_0 is $\bar{1}$ or $\bar{3}$ and each a_i is $\bar{0}$ or $\bar{2}$ for $i \geq 1$. However this implies that f_j is a unit in $\mathbb{Z}_4[X]$, a contradiction that we can write X^n as the product of m atoms where $m \geq n+1$. It follows that $L(X^n) = n$.

Now suppose $n > 1$ where n is even. In particular, we can factor $X^n = (X^{n/2} + \bar{2})(X^{n/2} + \bar{2})$, so $l(X^n) \leq 2$. If $l(X^n) = 1$, then X^n is irreducible in $\mathbb{Z}_4[X]$ where $n \neq 1$, a contradiction. So $l(X^n) = 2$.

Next suppose $n > 1$ where n is odd. We can factor $X^n = X(X^{n/2} + \bar{2})(X^{n/2} + \bar{2})$ so $l(X^n) \leq 3$. Since $n \neq 1$, X^n is not irreducible, so $l(X^n) \neq 1$. Suppose $l(X^n) = 2$. Then $X^n = fg$ with f, g irreducible in $\mathbb{Z}_4[X]$ where $f = a_0 + a_1X + \cdots + a_sX^s$ and $g = b_0 + b_1X + \cdots + b_tX^t$. Note that $a_0b_0 = \bar{0}$, so either $a_0 = b_0 = \bar{2}$, $a_0 = b_0 = \bar{0}$, or just one of a_0 or b_0 is zero. We note that each of these possibilities leads to a contradiction and thus $l(X^n) \neq 2$ which implies $l(X^n) = 3$ as desired. Each case is considered below:

<u>Case 1</u>: One of a_0 and b_0 is zero or $a_0 = b_0 = \bar{0}$

If just one of a_0 or b_0 is $\bar{0}$, say a_0, we have that f is divisible by X. Since we assume f is irreducible we must have $f = uX$, where u is a unit in $\mathbb{Z}_4[X]$. Then $X^n = fg = uXg$ implies $X^{n-1} = ug$ where g is irreducible. So X^{n-1} is irreducible for $n - 1 \geq 2$, a contradiction. If $a_0 = b_0 = \bar{0}$ then both f and g are both divisible by X, and a similar argument gives a contradiction.

<u>Case 2</u>: $a_0 = b_0 = \bar{2}$

Let $X^n = fg$ where $n \geq 3$ is odd and $a_0 = b_0 = \bar{2}$ where a_0 and b_0 are the constant terms of f and g respectively. Then we have

$$\begin{aligned} X^n &= fg \\ &= (2 + a_1X + \cdots + a_sX^s)(2 + b_1X + \cdots + b_tX^t) \\ &= c_1X + c_2X^2 + \cdots + c_nX^n + \cdots + c_{s+t}X^{s+t} \end{aligned}$$

where $c_n = \bar{1}$, and $c_i = \bar{0}$ for $i \neq n$. If we consider the homomorphic image of X^n in $\mathbb{Z}_2[X]$ then we can say $\overline{fg}$ has first factor $\overline{a_i}X^i$ with $i \geq 1$ and second factor $\overline{b_{n-i}}X^{n-i}$ with $1 \leq i < n - i$. Note that since a_i and b_{n-i} are units we can take them to be $\bar{1}$ without loss of generality. So we can write,

$$X^n = (A_1 + X^i + A_2X^{i+1})(B_1 + X^{n-i} + B_2X^{n-i+1})$$

where A_i and B_i are both in $\bar{2}\mathbb{Z}_4[X]$ and degree $A_1 \leq i-1$ and degree $B_1 \leq n-i-1$. Since A_i and B_i are in $\bar{2}\mathbb{Z}_4[X]$, each $A_iB_j = \bar{0}$. So we have,

$$\begin{aligned} X^n &= (A_1 + X^i + A_2X^{i+1})(B_1 + X^{n-i} + B_2X^{n-i+1}) \\ &= A_1B_1 + A_1X^{n-i} + A_1B_2X^{n-i+1} + B_1X^i \\ &+ X^n + A_2B_1X^{i+1} + B_2X^{n+1} + A_2X^{n+1} + A_2B_2X^{n+2} \\ &= \underbrace{A_1X^{n-i} + B_iX^i}_{=\bar{0}} + X^n + \underbrace{B_2X^{n+1} + A_2X^{n+1}}_{=\bar{0}}. \end{aligned}$$

Note that $A_1X^{n-i} + B_iX^i = \bar{0}$ since terms have degree less than n. Similarly, $B_2X^{n+1} + A_2X^{n+1}$ has degree greater than n so it is equal to $\bar{0}$. Then $\bar{0} = A_1X^{n-i} + B_1X^i =$

$X^i(A_1X^{n-2i} + B_1)$ implies that $A_1X^{n-2i} + B_1 = \bar{0}$. Since A_1X^{n-2i} has nonzero constant term, B_1 must have a zero constant term, thus $b_0 = \bar{0}$, a contradiction since we assumed $b_0 = \bar{2}$. □

A result related to Theorem 4.1.5 was independently done in [17].

4.2 When is $R[X]$ a Unique Factorization Ring?

We say an integral domain R is a unique factorization domain (UFD) if every nonzero nonunit can be written as a finite product of atoms and this factorization is unique up to order and associates. Here a nonzero nonuit x is an atom if $x = ab$ implies a or b is a unit and x and y are associates if they differ by a unit factor. It is easy to see that an integral domain is a UFD if and only if every nonzero nonunit is a product of principal primes. We can generalize these characterizations to commutative rings with zero divisors. In fact, given a commutative ring R, there are several types of unique factorization rings (UFRs) with zero divisors, established by various authors. Two important types of UFRs are those studied by Bouiver and Galovich, and those characterized by Fletcher. We will begin our discussion on UFRs in a polynomial ring with the former. We will also discuss different types of reduced UFRs, (α, β)-UFRs as described in Section 2.2, and weak unique factorization rings.

4.2.1 Bouvier-Galovich Unique Factorization Rings

Bouvier and Galovich both define a unique factorization ring to be a commutative ring R such that every nonzero nonunit is the product of irreducibles and if $0 \neq a_1 \cdots a_n = b_1 \cdots b_m$ with the a_i and b_i irreducible then $n = m$ and after a reordering a_i is associated to b_i for every i. However, they both have different definitions of irreducible and associate. Bouvier defined $a \in R$ to be irreducible if (a) is a maximal element of the set of proper principal ideals of R, which coincides with our definition of m-irreducible. Galovich defined $a \in R$ to be irreducible is $a = bc$ implies b or c is a unit, which is what we call very strongly irreducible. Bouvier's definition of associate agrees with ours, that is, $a \sim b$ implies $(a) = (b)$, while Galovich says $a \approx b$ if $a = ub$ for some $u \in U(R)$, or what we call strongly associate. Both Bouvier and Galovich also proved that their version of a unique factorization ring was characterized as being a UFD, quasi-local ring (R, M) with $M^2 = 0$, or a special principal ideal ring (SPIR). Note that a special principal ideal ring (SPIR) is a principal ideal ring with a unique prime ideal and that prime ideal is nilpotent. Thus these two notions of unique factorization ring coincide. This is stated in the theorem below.

Theorem 4.2.1. Given a commutative ring R, R is a Bouvier-Galovich unique factorization ring if and only if R satisfies one of the following:

1. R is a unique factorization domain,
2. R is a quasi-local ring with unique maximal ideal M where $M^2 = 0$, or
3. R is a special principal ideal ring.

Note that since a polynomial ring has infinitely many prime ideals, it cannot be a SPIR. A polynomial ring also cannot be quasi-local. It follows then from Theorem 5.11, that for a commutative ring R, $R[X]$ is a Bouvier-Galovich unique factorization ring if and only if $R[X]$ is a UFD. We prove this directly in the following theorem.

Theorem 4.2.2. Given a commutative ring R, $R[X]$ is a Bouvier-Galovich unique factorization ring if and only if $R[X]$ is a unique factorization domain.

Proof. ($\Longrightarrow$) Let $R[X]$ be a unique factorization ring and let $e \in R$ be idempotent. Suppose e is a nonzero nonunit, then $e = f_1 \cdots f_n$ where each f_i is irreducible. Then $f_1 \cdots f_n = e = e^2 = f_1^2 \cdots f_n^2$ so we have two factorizations of e into a product of irreducibles, a contradiction since $R[X]$ is a UFR. Thus e is a trivial idempotent and R is indecomposable. Now suppose there exists nonzero $a, b \in R$ such that $ab = 0$. Consider $X - a$, $X - b \in R[X]$. Since R is indecomposable, X is irreducible and so $X - a, X - b$ are also irreducible. To see this suppose $X - a = g(X)h(X)$ for some $g, h \in R[X]$. Then by a change of variables $X = g(X + a)h(X + a)$ which implies $X \sim g(X + a)$ or $X \sim h(X + a)$ since X is irreducible. Thus, $X - a \sim g(X)$ or $X - a \sim h(X)$. The same argument works for $X - b$, in fact $X - r$ is irreducible for any $r \in R$ when X is irreducible.

So now we have that,

$$(X - a)(X - b) = X^2 - (a + b)X + ab = X^2 - (a + b)X = X(X - (a + b))$$

are two factorizations of $X^2 - (a + b)X$ into the product of irreducibles in $R[X]$, a contradiction. So $R[X]$ a unique factorization ring implies that R is an integral domain, thus $R[X]$ is a unique factorization domain.

($\Longleftarrow$) If $R[X]$ is a unique factorization domain, then in particular, $R[X]$ is a unique factorization ring. $\square$

4.2.2 Fletcher Unique Factorization Rings

We now consider the second type of unique factorization ring, those characterized by Fletcher. His definition of associate agrees with ours while he defines $a \in R$ to be irreducible if $a = a_1 \cdots a_n$ implies $a \sim a_i$ for some i which is equivalent to our definition of irreducible. Next Fletcher defines the U-class of an element a to be the set $U(a) = \{r \in R \mid r(a) = (a)\}$ and a U-decomposition of an element to be a factorization such that $a = (a_1 \cdots a_k)(b_1 \cdots b_n)$ where a_i, b_j are irreducible, $a_i \in U(b_1 \cdots b_n)$ for every i and $b_j \notin U(b_1 \cdots \hat{b}_j \cdots b_n)$ for each j. Then a Fletcher unique factorization ring is a commutative ring R where every nonunit has a U-decomposition and if $(a_1 \cdots a_k)(b_1 \cdots b_n) = (a'_1 \cdots a'_{k'})(b'_1 \cdots b'_{n'})$ are two U-decompositions of a nonunit $a \in R$ then $n = n'$ and $b_i \sim b'_i$ after a reordering. The following theorem gives a characterization of Fletcher unique factorization rings.

Theorem 4.2.3. [21] For a commutative ring R, the following are equivalent:

1. R is a Fletcher UFR,
2. R is p-atomic, and
3. R is a finite direct product of UFDs and SPIRS.

Again, a polynomial ring cannot be a special principal ideal ring, so it follows from Theorem 4.2.3 that for a commutative ring R, $R[X]$ is a UFR in the sense of

Fletcher if and only if it is the finite direct product of UFDs. A direct proof of this was given by Anderson and Markanda in [9]. We provide another proof, but first we pay some attention to factorization of regular elements in a commutative ring R.

Definition 4.2.4. A commutative ring R is a *factorial ring* if every regular nonunit element of R is a product of (regular) irreducibles and this factorization is unique up to order and associates.

Lemma 4.2.5. Let R be a commutative ring. If $p_1, \ldots, p_n$ are regular primes and $a_1, \ldots, a_m$ are atoms such that $p_1 \cdots p_n = a_1 \cdots a_m$ then $n = m$ and after a reordering $p_i \sim a_i$ for every i.

Proof. Let $x \in R$ be regular. Suppose $x = p_1 \cdots p_n = a_1 \cdots a_m$ are two factorizations of x where the p_i are prime and a_j are irreducible. Then $a_1 \cdots a_m = p_1(p_2 \cdots p_n)$ so $a_1 \cdots a_m \in (p_1)$. Since p_1 is prime, one of the a_i is in (p_1), say a_1. Then $a_1 = r_1 p_1$ for some $r_1 \in R$. Note that a_1 is irreducible, so $a_1 \sim r_1$ or $a_1 \sim p_1$. If $a_1 \sim r_1$ then $a_1 \mid r_1$ gives $r_1 = a_1 s_1$ for some $s_1 \in R$. So $a_1 = r_1 p_1 = s_1 a_1 p_1$, and since a_1 is regular (because we assume x is regular), $a_1 = (s_1 p_1) a_1$ implies that $1 = s_1 p_1$. So p_1 is a unit, a contradiction since p_1 is prime. Thus $a_1 \sim p_1$ and r_1 is a unit.

Now we have $r_1 p_1 a_2 \cdots a_m = p_1 p_2 \cdots p_n$ so we can cancel p_1 from both sides to get $r_1 a_2 \cdots a_m = p_2 \cdots p_n$. Then $r_1 a_2 \cdots a_m \in (p_2)$ and one of the a_i is in (p_2) since p_2 is prime, say a_2. Then by a similar argument, $a_2 = r_2 p_2$ implies $a_2 \sim p_2$. We can continue in this manner until we see that $p_i \sim a_i$ for every i and $n = m$. $\square$

Note that Lemma 4.2.5 implies that if a regular element of R is a product of primes then this factorization is unique up to order and associates. We now characterize when a polynomial ring is a unique factorization ring in the sense of Fletcher.

Theorem 4.2.6. For a commutative ring R, the following are equivalent:

1. $R[X]$ is a Fletcher UFR,
2. $R[X]$ is p-atomic,
3. R is finite direct product of UFDs,
4. $R[X]$ is factorial, and
5. every regular element of $R[X]$ is a product of principal primes.

Proof. Note that from Fletcher's characterization of UFR's we have (1) $\Longleftrightarrow$ (3) and (1) $\Longleftrightarrow$ (2). However we will show the result from first principles.

(1) $\Longrightarrow$ (4) Let $R[X]$ be a Fletcher UFR and let $f \in R[X]$ be regular. Then a U-decomposition of f is of the form $()g_1 \cdots g_n$, that is, it has no irrelevant factors. Thus $f = g_1 \cdots g_n$ is just a factorization of f into atoms. Suppose there exist another factorization of f into irreducibles, say $f = g_1 \cdots g_n = h_1 \cdots h_m$. Then $()h_1 \cdots h_m$ is a U-decomposition of f. Since $R[X]$ is a Fletcher UFR, $()g_1 \cdots g_n = ()h_1 \cdots h_m$ gives $n = m$ and after a reordering $g_i \sim h_i$ for every i. It follows that $R[X]$ is factorial.

(4) $\Longrightarrow$ (3) Let $R[X]$ be factorial, then the regular elements of $R[X]$ have unique factorization into the product of regular irreducibles. In particular, since X is regular we can write it as the product of n atoms. By Theorem 4.1.8, this implies R is the finite direct product of n indecomposable rings, say $R \cong R_1 \times \cdots \times R_n$. Then $R[X] \cong R_1[X] \times \cdots R_n[X]$, since $R[X]$ is factorial, each $R_i[X]$ is factorial. Consider

$R_i[X]$ for arbitrary i. Since R_i is indecomposable, by Theorem 4.1.7, X is irreducible in $R_i[X]$. Suppose there exist $a, b \in R_i$ such that $ab = 0$ with a and b nonzero. Consider $X - a, X - b \in R[X]$. Since X is irreducible, $X - a$ and $X - b$ are also irreducible by remarks in Theorem 4.2.2. Then

$$(X - a)(X - b) = X^2 - (a + b)X + ab = X(X - (a + b))$$

are two factorizations of $X^2-(a+b)X$ into the product of regular irreducible elements, a contradiction. So each R_i is an integral domain. Thus $R[X]$ is the finite direct product of UFDs.

(3) $\Longrightarrow$ (1) Suppose R is the finite direct product of UFDs, say $R \cong R_1 \times \cdots \times R_n$. Then $R[X] \cong R_1[X] \times \cdots R_n[X]$ is a direct product of UFDs. Then in each $R_i[X]$, a U-decomposition of each nonunit has no irrelevant factors. Thus if $f = ()g_1 \cdots g_n = ()h_1 \cdots h_m$, we have $n = m$ and $g_i \sim h_i$ for every i. Since the direct product of Fletcher UFRs is a Fletcher UFR it follows that $R[X]$ is a Fletcher UFR.

(3) $\Longrightarrow$ (2) Suppose R is a finite direct product of UFDs, say $R \cong R_1 \times \cdots \times R_n$. Then $R[X] \cong R_1[X] \times \cdots \times R_n[X]$. Since each R_i is a UFD, each $R_i[X]$ is a UFD, thus each $R_i[X]$ is p-atomic. It follows that $R[X]$ is p-atomic.

(2) $\Longrightarrow$ (5) Let $R[X]$ be p-atomic, then every nonzero nonunit element of $R[X]$ is a finite product of prime elements. In particular, every regular element in $R[X]$ is a product of principal primes.

(5) $\Longrightarrow$ (4) Suppose every regular element of $R[X]$ is a product of principal primes. Since prime implies irreducible we have every regular element factors into the product irreducible principal ideals. By Lemma 4.2.5 this factorization is unique up to order

and associates, so $R[X]$ is factorial. □

4.2.3 (α, β)-Unique Factorization Rings

Recall from Section 2.2 that we defined an (α, β)-unique factorization ring R in the following way. Let $\alpha \in \{$atomic, strongly atomic, very strongly atomic, m-atomic, p-atomic$\}$ and $\beta \in \{$isomorphic, strongly isomorphic, very strongly isomorphic$\}$. Then a ring R is an (α, β)-UFR if (1) R is α and (2) any two factorizations of a nonzero, nonunit element into irreducible elements of the type used to define α are β. Thus a Bouvier unique factorization ring as a (m-atomic, isomorphic)-UFR. We also have that a Galovich unique factorization ring is a (very strongly atomic, strongly isomorphic)-UFR. Recall from 2.2 that R is présimplifiable for any choice of α and β above except for $\alpha = p$-atomic, so all forms of (α, β)-unique factorization are equivalent. Thus we call R a UFR if R is an (α, β)-UFR for all (α, β) except $\alpha = p$-atomic.

We can also consider when two factorizations of a nonzero nonunit are homomorhpic instead of isomorphic. Recall from [12] that two factorizations $a = a_1 \cdots a_n = b_1 \cdots b_m$ are *homomorphic* if for every $i \in \{1, \ldots, n\}$ there exists a $j \in \{1, \ldots, m\}$ with $a_i \sim b_j$ and for each $i \in \{1, \ldots, m\}$ there exists a $j \in \{1, \ldots, n\}$ so that $b_i \sim a_j$. We provide a characterization of an atomic polynomial ring $R[X]$ where any two factorizations of a nonzero nonunit are homomorphic. We will need the following lemmas:

Lemma 4.2.7. Let R be an integral domain. If any two factorizations of each nonzero nonunit element into atoms are homomorphic, then any two factorizations of each nonzero nonunit element are actually isomorphic.

Proof. Let $r = a_1 \cdots a_n = b_1 \cdots b_m$ be two factorizations of r into a product of atoms. Since atomic factorizations are homomorphic, a_1 must be associated to some b_i, after a reordering, we can say b_1. So we can cancel a_1 and b_1 from both sides (with possibly some unit u_1 left on the right hand side.) Then these two subfactorizations are also homomorphic, so we must have a_2 associated to some b_i, say b_2, and again we can cancel a_2 and b_2 from both sides (with possibly some unit u_2 left on the right hand side.) If we continue in this way we see that $n = m$. Along the way we have also shown that after reordering, a_i and b_i are associates, so the factorizations are actually isomorphic. □

Lemma 4.2.8. Let R be commutative. For $a, b \in R$ and $X - a, X - b \in R[X]$, $X - a \sim X - b$ if and only if $a = b$.

Proof. ($\Longrightarrow$) Suppose $X - a \sim X - b$. Then $X - a = (X - b)f(X)$ for some $f \in R[X]$. Then if we let $X = b$ we have $b - a = (b - b)f(b) = 0 \implies b = a$.

($\Longleftarrow$) Clear. □

Theorem 4.2.9. For a commutative ring R, the following are equivalent:

1. R, equivalently, $R[X]$ is a UFD,
2. $R[X]$ is a UFR,
3. $R[X]$ is indecomposable, p-atomic,

4. $R[X]$ is indecomposable, atomic, and any two factorizations of a nonzero nonunit are homomorphic, and

5. $R[X]$ indecomposable and factorial.

Proof. Clearly (1) $\implies$ (2). For (2) $\implies$ (4) note that since $R[X]$ is UFR it has no nontrivial idempotents, thus $R[X]$ is indecomposable. Also, since $R[X]$ is an (α, β)-UFR, $R[X]$ is atomic and any two factorizations of a nonzero nonunit are isomorphic, thus homomorphic. For (4) $\implies$ (1) suppose R is not an integral domain. Then there exist nonzero $a, b \in R$ with $ab = 0$. Since $R[X]$ is indecomposable, X is irreducible. So $X - r$ is irreducible for any $r \in R$. Then

$$(X - a)(X - b) = X^2 - (a + b)X = X(X - (a + b))$$

are two factorizations of $X^2 - (a + b)X$. Since any two factorizations of a nonzero nonunit are homomorphic, then in particular $X \sim X - a$ or $X \sim X - b$. By Lemma 4.2.8 if $X \sim X - a$ then $a = 0$, a contradiction since we assume a is nonzero. Similarly we cannot have $X \sim X - b$. So R is an integral domain, thus $R[X]$ is an integral domain. Suppose $f = f_1 \cdots f_n = g_1 \cdots g_m$ are two atomic factorization of f. By hypothesis they are homomorphic. Then by Lemma 4.2.7 $f_1 \cdots f_n$ and $g_1 \cdots g_m$ are two isomorphic factorizations of f. Thus $R[X]$ is an (atomic, isomorphic)-unique factorization ring and since R is a domain, $R[X]$ is a UFD.

Note that it is clear that (1) $\implies$ (3). For (3) $\implies$ (5) since $R[X]$ is p-atomic, any regular element has a factorization into primes. By Lemma 4.2.5 this factorization is unique up to order and associates, thus $R[X]$ is factorial. Then we have that

$(3), (5) \implies (1)$ since by Theorem 4.2.6 $R[X]$ p-atomic implies R is the finite direct product of UFDs. But $R[X]$ is indecomposable, so R is indecomposable. Thus R, equivalently, $R[X]$ is a UFD. Similarly, $R[X]$ factorial and indecomposable implies $R[X]$ is a UFD. □

4.2.4 μ-Reduced and Reduced Unique Factorization Rings

Next we will characterize another type of unique factorization ring. In [7], Anderson, Chun, and Valdes-Leon defined various types of reduced factorizations of a nonzero nonunit in a commutative ring R and discussed unique factorization properties with respect to these reduced factorizations. In this section we will define what they called μ-reduced and reduced factorizations and the corresponding unique factorization rings that arise with respect to each. Then we will discuss these different type of unique factorization rings in the context of polynomial rings.

Definition 4.2.10. In a commutative ring R, a *μ-factorization* of a nonunit $a \in R$ is a factorization $a = \lambda a_1 \cdots a_n$ where $n \geq 1$, $\lambda \in U(R)$, and each $a_i \in R - U(R)$. A μ-factorization is *μ-reduced* (respectively *strongly μ-reduced*) if $a \neq \lambda' a_1 \cdots \hat{a_i} \cdots a_n$ for any $\lambda' \in U(R)$ and any i (respectively $a \neq \lambda' a_1 \ldots \hat{a_{i_1}} \cdots \hat{a_{i_j}} \cdots a_n$ for any $\lambda' \in U(R)$ and any nonempty proper subset $\{i_1, \ldots, i_j\} \subsetneq \{1, \ldots, n\}$).

Definition 4.2.11. In a commutative ring R, a factorization $a = a_1 \cdots a_n$ of a nonunit $a \in R$ is *reduced* (respectively *strongly reduced*) if $a \neq a_1 \cdots \hat{a_i} \cdots a_n$ for any $i \in$

$\{1, \ldots, n\}$ (respectively $a \neq a_1 \cdots \hat{a_{i_1}} \cdots \hat{a_{i_j}} \cdots a_n$ for any nonempty property subset $\{i_1, \ldots, i_j\} \subsetneq \{1, \ldots, n\}$).

Definition 4.2.12. In a commutative ring R, a μ-factorization $a = \lambda a_1 \cdots a_n$ is *reduced* (respectively *strongly reduced*) if $a \neq \lambda a_1 \cdots \hat{a}_i \cdots a_n$ for any i (respectively $a \neq \lambda a_1 \cdots \hat{a_{i_1}} \cdots \hat{a_{i_j}} \cdots a_n$ for any nonemptry proper subset $\{i_1, \ldots, i_j\} \subsetneq \{1, \ldots, n\}$).

Note that a reduced and strongly reduced factorization of a nonunit $a \in R$ is just a μ-reduced factorization where $\lambda = 1$. Thus any μ-factorization of a can be simplified to a μ-reduced, strongly μ-reduced, reduced, or strongly reduced factorization of a in a non-unique way. We have the following implications for the different types of reduced factorizations, none of which can be reversed.

$$\begin{array}{ccc} \text{strongly } \mu\text{-reduced} & \Longrightarrow & \text{strongly reduced} \\ \Downarrow & & \Downarrow \\ \mu\text{-reduced} & \Longrightarrow & \text{reduced} \end{array}$$

Each type of reduced factorization above gives rise to a type of unique factorization ring. In particular, we can define a strongly μ-reduced, μ-reduced, strongly reduced, and reduced unique factorization ring. We seek to characterize when a polynomial ring is a unique factorization ring with respect to each type of reduced factorization. We begin with a discussion on strongly μ-reduced and reduced UFRs.

Definition 4.2.13. A commutative ring R is a *strongly μ-reduced* (respectively *μ-reduced*) *unique factorization ring* if:

1. R is atomic and

2. for each nonzero nonunit $a \in R$, whenever $a = \lambda_1 a_1 \cdots a_n = \lambda_2 b_1 \cdots b_m$ are two strongly μ-reduced (respectively μ-reduced) atomic μ-factorizations for a, we have $n = m$ and after a re-ordering, $a_i \sim b_i$ for $i = 1, \ldots, n$.

In [7] Anderson, Chun, and Valdes-Leon give a characterization of strongly μ-reduced and μ-reduced UFRs. We include this in the following theorem.

Theorem 4.2.14. [7, Theorem 3.3] For a commutative ring R the following conditions are equivalent:

1. R is a strongly μ-reduced UFR.
2. R is a μ-reduced UFR.
3. R is a finite direct product of UFDs and SPIRs.

Now we characterize when a polynomial ring is a strongly μ-reduced and μ-reduced UFR. It is clear from above that since a polynomial ring cannot be an SPIR, it is a strongly μ-reduced or μ-reduced UFR if and only if it is a finite direct product of UFDs. We prove this from first principles in Theorem 4.2.16. First we need a lemma.

Lemma 4.2.15. Let R be a commutative ring. If R is the finite direct product of strongly μ-reduced (respectively μ-reduced) UFRs, then R is a strongly μ-reduced (respectively μ-reduced) UFR.

Proof. Let $R = R_1 \times R_2$ where R_1 and R_2 are both strongly μ-reduced unique factorization rings. Let $(x, y) \in R$ be a nonzero nonunit. Then x and y are not both units in R_1 and R_2. Suppose neither is a unit. Then there exist two strongly μ-reduced factorizations of x and y in R_1 and R_2 respectively, say $x = \lambda_x x_1 \cdots x_n$ and $y = \lambda_y y_1 \cdots y_m$ with the x_i and y_j atoms. Then

$$(x, y) = (\lambda_x, \lambda_y)(x_1, 1) \cdots (x_n, 1)(1, y_1) \cdots (1, y_m)$$

is a strongly μ-reduced factorization of (x, y) in R with each $(x_i, 1)$ and $(1, y_j)$ an atom. Similarly, if one of x or y is a unit, say y, then

$$(x, y) = (\lambda_x, y)(x_1, 1) \cdots (x_n, 1)$$

gives a strongly μ-reduced factorization of (x, y). Thus R is a atomic.

Now suppose there exist two strongly μ-reduced factorizations of (x, y) into atoms, say

$$\begin{aligned}(x, y) &= (\lambda_x, \lambda_y)(x_1, 1) \cdots (x_n, 1)(1, y_1) \cdots (1, y_m)\\ &= (\lambda'_x, \lambda'_y)(x'_1, 1) \cdots (x'_{n'}, 1)(1, y'_1) \cdots (1, y'_{m'}).\end{aligned}$$

Then $x = \lambda_x x_1 \cdots x_n = \lambda'_x x'_1 \cdots x'_{n'}$ are two strongly μ-reduced factorizations of x in R_1. Since R_1 is a strongly μ-reduced UFR then $n = n'$ and $x_i \sim x'_{i'}$ after a possible reordering. Similarly $y = \lambda_y y_1 \cdots y_m = \lambda'_y y'_1 \cdots y'_{m'}$ in R_2 which is a strongly μ-reduced UFR so $m = m'$ and the $y_j \sim y'_{j'}$ after a possible reordering. Thus, in the two strongly μ-reduced factorizations of (x, y) in R we have that $n + m = n' + m'$, $(x_i, 1) \sim (x'_{i'}, 1)$ and $(1, y_j) \sim (1, y'_{j'})$ after a possible reordering. It follows that R is a strongly μ-reduced UFR. Using the same argument, by induction we have that the finite direct product of strongly μ-reduced UFRs is a strongly μ-reduced UFR. A similar argument shows that the finite direct product of μ-reduced UFRs is itself a μ-reduced UFR. $\square$

Theorem 4.2.16. For a commutative ring R, the following are equivalent:

1. $R[X]$ is a strongly μ-reduced UFR,
2. $R[X]$ is a μ-reduced UFR, and
3. R is a finite direct product of UFDs.

Proof. (1) $\Longrightarrow$ (2) Clear since strongly μ-reduced implies μ-reduced.

(2) $\Longrightarrow$ (3) Since $R[X]$ is a μ-reduced UFR, we know that X has a μ-reduced factorization into atoms, say $X = \lambda_f f_1 \cdots f_n$ where $\lambda \in U(R[X])$. Since X is regular, μ-reduced atomic factorizations are the same as atomic factorizaitons, thus we can say X is the product of n atoms. So $R = R_1 \times \cdots \times R_n$ is the finite direct product of n indecomposable rings, and $R[X] = R_1[X] \times \cdots R_n[X]$ with X irreducible in $R_i[X]$ for every i. Note that each $R_i[X]$ is a μ-reduced UFR. Now consider $(X) \subseteq R_j[X]$ for some j. Suppose $0 \neq fg \in (X)$ with $f, g \in R_j[X]$. Then $fg = Xh$ for some $h \in R_j[X]$. If we factor, f, g, and h into atoms, then we have two μ-reduced atomic factorizations of fg, so X is associated to one of the irreducible factors of f or g, say f. Then $f \in (X)$. Now suppose $0 = fg \in (X)$. Then in particular, $f \in Z(R[X])$ so there exists a nonzero $c \in R$ such that $cf = 0$. So we have

$$0 \neq fX = fX + fc = f(X + c)$$

and $f(X + c) = fX \in (X)$. From the previous argument if we take $g = X + c$ we have $f \in (X)$ or $X + c \in (X)$. If $X + c \in (X)$ then $c \in (X)$, a contradiction since we assumed c is nonzero. So $f \in (X)$. Thus (X) is a prime ideal. So R_j is an integral domain and hence $R_j[X]$ is a UFD. Since $R_j[X]$ was arbitrary, it follows that $R[X]$ is the finite direct product of UFDs.

(3) $\Longrightarrow$ (1) Suppose $R[X] = R_1[X] \times \cdots \times R_n[X]$ where the R_i are UFDs for every i. Then since each $R_i[X]$ is a UFD, in particular each $R_i[X]$ is atomic and strongly μ-reduced, and thus is a strongly μ-reduced UFR. Since the finite direct product of strongly μ-reduced UFRs is a strongly μ-reduced UFR by Lemma 4.2.15, it follows that $R[X]$ is a strongly μ-reduced UFR. $\square$

Now we move our discussion to strongly reduced and reduced unique factorization rings. First we provide the definitions given in [7] and include a characterization of these types of UFRs. Then we provide a characterization for when a polynomial ring is a strongly reduced or reduced unique factorization ring.

Definition 4.2.17. A commutative ring R is a *strongly reduced* (respectively *reduced*) unique factorization ring if:

1. R is atomic, (and hence every nonunit of R has a strongly reduced (respectively reduced) factorization into the product of atoms) and
2. for every nonunit $a \in R$ with a strongly reduced (respectively reduced) factorization, $a = a_1 \cdots a_n$, if there exists another strongly reduced (respectively reduced) factorization $a = b_1 \cdots b_m$, then $n = m$ and after a reordering $a_i \sim b_i$ for $i = 1, \ldots, n$.

The following theorem from [7] characterizes when a commutative ring R is a reduced or strongly reduced unique factorization ring.

Theorem 4.2.18. [7, Theorem 3.4] For a commutative ring R, the following are equivalent:

1. R is a reduced UFR.
2. R is a strongly reduced UFR.
3. R is either (a) a UFD, (b) an SPIR, or (c) a finite direct product $D_1 \times \cdots \times D_n$ $(n \geq 2)$ where each D_i is a UFD (possibly a field) with $U(D_i) = \{1\}$.

Now we provide the following characterization for a polynomial ring to be a strongly reduced and reduced UFR.

Theorem 4.2.19. For a commutative ring R, the following are equivalent:

1. $R[X]$ strongly reduced UFR,
2. $R[X]$ reduced UFR, and
3. R is a UFD or a finite direct product $D_1 \times \cdots \times D_n$ $(n \geq 2)$ and each D_i is a UFD (possibly a field) with group of units $U(D_i) = \{1\}$.

Proof. (1) $\Longrightarrow$ (2) Clear since strongly reduced implies reduced.

(2) $\Longrightarrow$ (3) Since $R[X]$ is a reduced UFR we have that X has a reduced factorization into atoms, say $X = f_1 \cdots f_n$. Note that since X is regular, a reduced atomic factorization is the same as an atomic factorization. So we have that $R = R_1 \times \cdots \times R_n$ is the finite direct product of n indecomposable rings. Then we can write $R[X] = R_1[X] \times \cdots \times R_n[X]$. Since each $R_i[X]$ is indecomposable, X is irreducible in each $R_i[X]$.

We now show that $XR_i[X]$ is a prime ideal. Suppose that $0 \neq fg \in (X)$ for some $f, g \in R_i[X]$. Then $fg = hX$ for some $h \in R_i[X]$ and we have $f = f_1 \cdots f_s$, $g = g_1 \cdots g_t$, and $h = h_1 \cdots c_r$ which are reduced factorizations of f, g, and h respectively. So we have $f_1 \cdots f_s g_1 \cdots g_t = h_1 \cdots h_r X$. Since $R_i[X]$ is a reduced unique factorization ring each h_i is associated to one of the $s+t$ atoms on the left hand side and X is associated to one of the f_i or g_j, say one of the f_i. Then $f \in (X)$. Now suppose $0 = fg \in (X)$. Then in particular, $f \in Z(R[X])$ so there exists a nonzero $c \in R$ such that $cf = 0$. So we have

$$0 \neq fX = fX + fc = f(X+c)$$

and $f(X+c) = fX \in (X)$. From the previous argument if we take $g = X + c$ we have $f \in (X)$ or $X + c \in (X)$. If $X + c \in (X)$ then $c \in (X)$, a contradiction since we assumed c is nonzero. So $f \in (X)$. Thus (X) is a prime ideal, so R_i is an integral domain and hence a UFD. It follows that $R[X]$ a reduced UFR implies R is the finite direct products of UFDs, or if $n = 1$, that R is a UFD.

Now suppose that $n > 1$ and $|U(R_j)| > 1$ for some j. Then there exists a $u \in U(R_j)$ with $u \neq 1$. Then there is a $v \in U(R_j)$ so that $uv = 1$. Note that each R_i an integral domain implies that 0 is prime in each R_i and thus irreducible. Then we have

$$(0,1,\ldots,1) = (0,1,\ldots,1,u,1,\ldots,1)(0,1,\ldots,1,v,1,\ldots,1) = (0,1,\ldots,1,\ldots,1)$$

are two reduced factorizations of $(0,1,\ldots,1)$ into the product of atoms, a contradiction. Thus $U(R_j) = \{1\}$.

(3) $\Longrightarrow$ (1) Suppose R is a UFD, then $R[X]$ is a UFD, in particular it is a strongly reduced UFR. We next need to show that if $D_1, \ldots, D_n$ are UFDs with $U(D_i) = \{1\}$, then $D_1[X] \times \cdots \times D_n[X]$ is a strongly reduced UFR. For simplification of notation we do the case $n = 2$. The general case is similar. Now suppose $R = R_1 \times R_2$ with each R_i a UFD, $U(R_1) = \{1\}$, and $U(R_2) = \{1\}$. Consider a nonunit $(f, g) \in R_1[X] \times R_2[X]$. Assume both f and g are nonunits in $R_1[X]$ and $R_2[X]$ respectively. The proof if one of f or g is a unit is similar. Note that there exist factorizations $f = f_1 \cdots f_n$ and $g = g_1 \cdots g_m$ into atoms in $R_1[X]$ and $R_2[X]$ respectively, so we have the following factorization of (f, g) into atoms in $R[X]$,

$$(f, g) = (f_1, 1) \cdots (f_n, 1)(1, g_1) \cdots (1, g_m).$$

Suppose that this is not a strongly reduced factorization of (f, g). Then we can write

$$(f, g) = (f_1, 1) \cdots \widehat{(f_{i_1}, 1)} \cdots \widehat{(f_{i_k}, 1)} \cdots (f_n, 1)(1, g_1) \cdots \widehat{(1, g_{j_1})} \cdots \widehat{(1, g_{j_k})} \cdots (1, g_m).$$

Then $f_1 \cdots \hat{f_{i_1}} \cdots \hat{f_{i_k}} \cdots f_n$ and $g_1 \cdots \hat{g_{j_1}} \cdots \hat{g_{j_l}} \cdots g_m$ are factorizations of f and g into atoms in $R_1[X]$ and $R_2[X]$ respectively. Then $f = f_1 \cdots f_n = f_1 \cdots \hat{f_{i_1}} \cdots \hat{f_{i_k}} \cdots f_n$ implies that $1 = f_{i_1} \cdots f_{i_k}$, so $f_{i_1}, \ldots, f_{i_k} \in U(R_1[X]) = \{1\}$. Similarly, we must have $g_{j_1}, \ldots, g_{j_l} \in U(R_2[X]) = \{1\}$. Thus we arrive at a contradiction since we assume that each $(f_i, 1)$ and $(1, g_j)$ is an atom in the factorization of (f, g). So every nonunit in $R[X]$ has a strongly reduced factorization into atoms.

Now suppose there are two strongly reduced factorizations of (f, g) into atoms,

$$(f, g) = (f_1, 1) \cdots (f_n, 1)(1, g_1) \cdots (1, g_m) = (f'_1, 1) \cdots (f'_{n'})(1, g'_1) \cdots (1, g'_{m'}).$$

Then $f_1 \cdots f_n = f'_1 \cdots f'_{n'}$ are two strongly reduced factorizations of f into atoms in $R_1[X]$. Since $R_1[X]$ is a domain, strongly reduced factorizations into atoms are the same as atomic factorizations. Thus $n = n'$ and $f_i \sim f'_{i'}$ after a possible reordering. Similarly, in $R_2[X]$ we have $m = m'$ and $g_j \sim g'_{j'}$ after a possible reordering. Then we have that $n+n' = m+m'$ in the factorization of $(f, g) \in R[X]$ and the $(f_i, 1) \sim (f'_{i'}, 1)$ and $(1, g_j) \sim (1, g'_{j'})$ after a possible reordering. It follows that $R[X]$ is a strongly reduced unique factorization ring. □

4.2.5 Weak Unique Factorization Rings

In [11] Anderson and Smith discuss weakly prime elements in a commutative ring R. They extend this notion to weakly prime ideals and gave a number of results concerning weakly prime ideals. Weakly prime elements were first discussed by Galovich, though he referred to them as just primes, and were recoined as "weakly" prime in a paper by Agargün, Anderson, and Valdes-Leon, *Unique factorization rings with zero divisors* [1]. We begin our discussion of weakly primes in the context of polynomial rings with the following definitions.

Definition 4.2.20. Let R be a commutative ring. A nonzero nonunit $p \in R$ is *weakly prime* if $p \mid ab \neq 0$ implies $p \mid a$ or $p \mid b$.

Definition 4.2.21. Let R be a commutative ring. A proper ideal P of R is *weakly prime ideal* if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$.

Note that every prime ideal is weakly prime, and the (0) ideal is weakly prime but not necessarily prime. For a nontrivial example, in [11] it is remarked that every proper ideal in a quasi-local ring (R, M) with $M^2 = 0$ is weakly prime. The goal of this section is to characterize weakly prime ideals in a polynomial ring $R[X]$ and to give a characterization of weak unique factorization rings in this context. To define a weak UFR we first need to define weakly homomorphic factorizations. Recall from [12] that two factorizations $a = a_1 \cdots a_n = b_1 \cdots b_m$ are *homomorphic* if for every $i \in \{1, \ldots, n\}$ there exists a $j \in \{1, \ldots, m\}$ with $a_i \sim b_j$ and for each $i \in \{1, \ldots, m\}$ there exists a $j \in \{1, \ldots, n\}$ so that $b_i \sim a_j$.

Definition 4.2.22. Two factorizations $a_1 \cdots a_n = b_1 \cdots b_m$ into atoms are *weakly homomorphic* if for each $i \in \{1, \ldots, n\}$, there exists $j \in \{1, \ldots, m\}$ such that $a_i \mid b_j$ and for each for each $i \in \{1, \ldots, m\}$ there exists $j \in \{1, \ldots, n\}$ such that $b_i \mid a_j$.

Definition 4.2.23. Let R be a commutative ring. Then R is a *weak unique factorization ring* if:

1. R is atomic, and
2. any two factorizations of a nonzero nonunit $a \in R$ into irreducibles are weakly homomorphic.

Note that a weak UFR is an (atomic, weakly homomorphic)-unique factorization ring. In [1] it is shown that a Fletcher UFR is a weak UFR and stated that a Bouvier-Galovich UFR is a weak UFR. One of the central results of [1] involves the following theorem and corollary.

Theorem 4.2.24. [1, Theorem 2.3] Let R be a commutative ring. Then the following are equivalent:

1. R is a weak UFR,
2. every nonzero nonunit of R is a product of weakly prime elements, and
3. R is atomic and each irreducible element of R is weakly prime.

Corollary 4.2.25. [1, Theorem 2.13] Let R be a commutative ring, then R is a weak UFR if and only if R is a finite direct product of UFDs and SPIRs or R is a quasilocal ring (R, M) with $M^2 = 0$.

Theorem 4.2.26 is clear from Theorem 4.2.24 but we restate this result below for a polynomial ring $R[X]$.

Theorem 4.2.26. Let R be commutative. Then the following are equivalent,

1. $R[X]$ is a weak UFR,
2. Every nonzero nonunit of $R[X]$ is a product of weakly primes,
3. $R[X]$ is atomic and each irreducible of $R[X]$ is weakly prime.

Lemma 4.2.27. If $R = R_1 \times R_2$ is a weak UFR, then each R_i is a weak UFR.

Proof. Let $x \in R_1$ be a nonzero nonunit, then $(x, 1) \in R$ is a nonzero nonunit. Since R is atomic, we can write $(x, 1) = (x_1, 1) \cdots (x_n, 1)$ as the product of atoms in R. But then we can write $x = x_1 \cdots x_n$ in R_1 so R_1 is atomic. Similarly, we can show that R_2 is atomic. Now suppose there exists two factorizations of a nonzero nonunit into atoms, say $x_1 \cdots x_k = a_1 \cdots a_l$. Then in R we have that $(x_1, 1) \cdots (x_k, 1) = (a_1, 1) \cdots (a_l, 1)$.

Since R is a weak UFR these two factorizations into atoms are weakly homomorphic. So for each $i \in \{1, \ldots, k\}$, there exists $j \in \{1, \ldots, l\}$ so that $(x_i, 1) \mid (a_j, 1)$ and for each $i \in \{1, \ldots, l\}$ there exists $j \in \{1, \ldots, k\}$ so that $(a_i, 1) \mid (x_j, 1)$. So we must have that for $i \in \{1, \ldots, k\}$, there exists $j \in \{1, \ldots, l\}$ so that $x_i \mid a_j$ and for each $i \in \{1, \ldots, l\}$ there exists $j \in \{1, \ldots, k\}$ so that $a_i \mid x_j$ in R_1. So the two factorizations are weakly homomorphic. A similar argument holds for any two factorizations of a nonzero nonunit into atoms in R_2. Thus if $R = R_1 \times R_2$ is a weak UFR then each R_i is a weak UFR. $\square$

Theorem 4.2.28. Let R be a commutative ring. Then $R[X]$ is a weak unique factorization ring if and only if R is the finite direct product of UFDs.

Proof. ($\Longrightarrow$) Let $R[X]$ be a weak UFR. Since X is a nonzero nounit, it is the product of atoms, say n of them. Then R is the finite direct product of n indecomposable rings where $R = R_1 \times \cdots \times R_n$. Thus $R[X] = R_1[X] \times \cdots \times R_n[X]$ with X irreducible in $R_i[X]$ for every i. Note that we can extend the result in Lemma 4.2.27 by induction to see that each $R_i[X]$ is a weak UFR since $R[X]$ is a weak UFR. Consider the ideal generated by X, (X), in some $R_j[X]$. We want to show that (X) is prime, for then it follows that $R_j[X]$ is an integral domain and hence a UFD. First let $0 \neq fg \in (X)$ with $f, g \in R_j[X]$. Then $fg = Xh$ for some $h \in R_j[X]$. We can factor f, g, and h into atoms since $R_j[X]$ is a weak UFR and thus atomic. Then we have two factorizations into atoms that must be weakly homomorphic, so X divides one of the irreducible factors of f or g, say f. Then $f \in (X)$, so (X) is a weakly prime ideal. Now suppose $0 = fg \in (X)$. If f is zero then $f \in (X)$, similarly, $g \in (X)$ if g is zero. So suppose

f, g are both nonzero. In particular, since $f \in Z(R[X])$ there exists $c \in R$ nonzero so that $fc = 0$. Note that since f is nonzero, fX is nonzero. Then

$$0 \neq fX = fX + fc = f(X + c).$$

Since (X) is weakly prime, $0 \neq f(X + c) = fX \in (X)$ implies that $f \in (X)$ or $X + c \in (X)$. If $X + c \in (X)$ then $c \in (X)$, a contradiction since we assume c is nonzero. So we must have $f \in (X)$. It follows that (X) is a prime ideal. Since $R_j[X]$ was arbitrary, it follows that $R[X]$ is the finite direct product of UFDs.

($\Longleftarrow$) Suppose R is the finite direct product of UFDs, say $R = R_1 \times \cdots \times R_n$, then $R[X] = R_1[X] \times \cdots \times R_n[X]$ is the finite direct product of UFDs. By Theorem 4.2.6 $R[X]$ is p-atomic. So every (nonzero) nonunit is a product of weakly primes and hence $R[X]$ is a weak UFR. $\square$

Note that in the proof of Theorem 4.2.28 we see that X is weakly prime if and only if X is prime. In [11, Theorem 1] Anderson and Smith show that for a weakly prime ideal P, either $P^2 = 0$ or P is prime. So if p is a weakly prime element with $p^2 \neq 0$, then p is prime. Since $X^2 \neq 0$ it also follows from this result that if X is weakly prime then X is prime. We can also conclude that X is a product of weakly prime elements if and only if it is a product of prime elements. However, X is a product of prime elements if and only if R is a direct product of integral domains. We prove these facts formally below.

Corollary 4.2.29. X is a product of weakly prime elements if and only if it is a product of prime elements.

Proof. ($\Longrightarrow$) if $X = p_1 \cdots p_n$ where the p_i are weakly prime, then for each p_i we have either $p_i^2 = 0$ or p_i is prime by the previously mentioned result in [11]. If $p_j^2 = 0$ for some p_j, then we have $X^2 = p_1^2 \cdots p_j^2 \cdots p_n^2 = 0$, a contradiction. So each p_i is prime.

($\Longleftarrow$) Clear. $\square$

Theorem 4.2.30. X is a product of primes if and only if R is a finite direct product of domains.

Proof. ($\Longrightarrow$) Suppose X is a product of primes, say $X = p_1 \cdots p_n$, then we have that R is the finite direct product of n indecomposable rings. So $R \cong R_1 \times \cdots \times R_n$. Then $R[X] \cong R_1[X] \times \cdots \times R_n[X]$. So $X \mapsto (X, \ldots, X) = X_1 \cdots X_n$ where each X_i has X in the i^{th} coordinate and a unit in all others. Then since X is the product of primes, by Lemma 4.2.5, each X_i is prime. Since primes in the direct product are prime in one coordinate and a unit in all others, each X_i prime implies X is prime in each $R_i[X]$. So each $R_i[X]$ is an integral domain, thus R is the finite direct product of domains.

($\Longleftarrow$) Now assume R is the finite direct product of domains, say $R \cong R_1 \times \cdots \times R_n$. Then $R[X] \cong R_1[X] \times \cdots \times R_n[X]$ is the finite direct product of domains. Then $X \mapsto (X, \ldots, X) = X_1 \ldots X_n$ where each X_i has X in the i^{th} coordinate and 1 in all the others. Since each R_i is a domain, X is prime in each $R_i[X]$. Since primes in a direct product are prime in one coordinate and a unit in all others, each X_i is prime. Thus X is the product of primes. $\square$

Theorem 4.2.31. Let P be a prime ideal of R with $P^2 = 0$. If Q is P-primary, then Q is weakly prime. Thus $Q[X]$ is also a weakly prime ideal of $R[X]$.

Proof. Suppose Q is not weakly prime. Then there is some $0 \neq ab \in Q$ with both $a, b \notin Q$. Since Q is P-primary, there must be positive integers n, m so that $a^n, b^m \in Q$. Thus $a, b \in rad(Q) = P$, and $a, b \in P$ implies $ab = 0$, a contradiction. So Q is weakly prime. Then $Q[X]$ is a $P[X]$-primary ideal in $R[X]$ and it follows that $Q[X]$ is also weakly prime. $\square$

Theorem 4.2.32. Let (R, M) be a zero-dimensional quasilocal ring. Suppose that $Q \subseteq M[X]$ is weakly prime. Then Q is $M[X]$-primary.

Proof. First we show that Q is primary. Since Q is weakly prime either Q is prime or $Q^2 = 0$. If Q is prime we are done, so assume $Q^2 = 0$. Let $fg \in Q$. If $fg \neq 0$, then Q weakly prime implies that $f \in Q$ or $g \in Q$. So, suppose $0 = fg \in Q$ with both $f, g \notin Q$. Since $Q \subseteq M[X]$ this implies $fg \in M[X]$ so $f \in M[X]$ or $g \in M[X]$ since $M[X]$ is prime, say $f \in M[X]$. Since (R, M) is a zero-dimensional quasilocal ring, M is the unique prime ideal of R, and every element of M is nilpotent. Thus $f \in M[X]$ implies every coefficient of f is nilpotent, so $f \in Nil(R[X])$. Then $f^n = 0 \in Q$ for some positive integer n. It follows that Q is primary.

Next, note that since $Q \subseteq M[X]$, we have $rad(Q) \subseteq rad(M[X]) = M[X]$. Assume that $radQ \subsetneq M[X]$. Then there is some $f \in M[X]$ with $f \notin rad(Q)$. But by the remarks above $f \in M[X]$ implies $f \in Nil(R[X])$ so $f^m = 0 \in Q$ for some positive integer m. Thus $f \in rad(Q)$, a contradiction. So $rad(Q) = M[X]$. It follows that Q is $M[X]$-primary. $\square$

Corollary 4.2.33. Let (R, M) be a quasilocal ring with $M^2 = 0$. Let $Q \subseteq M[X]$ be an ideal of $R[X]$. Then Q is weakly prime if and only if Q is $M[X]$-primary.

Proof. ($\Longrightarrow$) First we show that Q is primary. If Q is prime we are done. So suppose $Q^2 = 0$. Let $fg \in Q$. If $0 \neq fg$ then Q weakly prime implies that $f \in Q$ or $g \in Q$. So assume $0 = fg$. If one of both of f, g are in Q we are done, so suppose $f, g \notin Q$. Then $fg \in M[X] \implies f \in M[X]$ or $g \in M[X]$ since $M[X]$ is prime, say $f \in M[X]$. Then $f^2 = 0 \in Q$. Similarly, if $g \in M[X]$, $g^2 = 0 \in Q$. So Q is primary.

Now since $Q \subseteq M[X]$ then $rad(Q) \subseteq rad(M[X]) = M[X]$. Suppose $rad(Q) \subsetneq M[X]$. Then there exists an $f \in M[X]$ such that no power of f lies in Q. But $f \in M[X]$ implies that $f^2 = 0 \in Q$, a contradiction, so $rad(Q) = M[X]$. It follows that Q is $M[X]$-primary.

($\Longleftarrow$) Since $M^2 = 0$ implies $(M[X])^2 = 0$ it follows from Theorem 4.2.31 that Q is weakly prime.

$\square$

CHAPTER 5
INDECOMPOSABLE POLYNOMIALS

5.1 Types of Indecomposable Polynomials

In this section we discuss indecomposable polynomials in $R[X]$ where R is an arbitrary commutative ring. We begin with indecomposable polynomials in an integral domain, and consider how these notions extend to a polynomial ring with zero divisors. If R is an integral domain, then we have several equivalent conditions for an indecomposable polynomial given in the theorem below.

Theorem 5.1.1. Let R be an integral domain. For a polynomial $f \in R[X]$ the following are equivalent:

1. $f \neq gh$ with $\deg g, h \geq 1$,
2. $f = gh$ implies that $\deg g \leq 0$ or $\deg h \leq 0$, and
3. $f = gh$ implies that $\deg f = \deg g$ or $\deg f = \deg h$

Proof. (1) $\Longrightarrow$ (2): Let $f \in R[X]$ and suppose $f = gh$. Then by (1) we cannot have both $\deg g$ and $\deg h$ greater than or equal to 1. So we must have one of g or h with $\deg \leq 0$. Note that we assume $\deg 0 = -\infty$.

(2) $\Longrightarrow$ (3): Suppose $f = gh$. Then by (2) $\deg g \leq 0$ or $\deg h \leq 0$. Say $\deg g \leq 0$. If $\deg g < 0$ then $g = 0$ which implies $f = 0$, so $\deg g = \deg f = -\infty$. If $\deg g = 0$, then $g = a$ for some $a \in R$, then $\deg f = \deg ah = \deg h$.

(3) $\Longrightarrow$ (1): Suppose $f = gh$ with $\deg g, h \geq 1$. Then by (3) we have $\deg f = \deg g$

or $\deg f = \deg h$. Say $\deg f = \deg g$, then we have

$$\begin{aligned}\deg f &= \deg(gh)\\ &= \deg g + \deg h\\ &= \deg f + \deg h\\ &\geq \deg f + 1\end{aligned}$$

a contradiction, so $f \neq gh$ with $\deg g, h \geq 1$. □

We can generalize conditions (1) and (2) in the theorem above to a polynomial ring with zero divisors with the following definition.

Definition 5.1.2. Let R be a commutative ring. Then $f \in R[X]$ is *regularly decomposable* if it can be written as a product of two polynomials of positive degree. If f cannot be written as a product of two polynomials of positive degree we say f is *regularly indecomposable*.

It is clear that the definition for a regularly indecomposable polynomial is equivalent to conditions (1) and (2) in Theorem 5.1.1. In the remarks that follow, we discuss some properties of regularly indecomposable and regularly decomposable polynomials in $R[X]$. In particular, we show that if R is a reduced ring, then f is very strongly irreducible if it is regularly indecomposable, and that for any ring R if f has positive degree and is a zero divisor it is regularly decomposable. In fact every polynomial of positive degree at least 1 is regularly decomposable if R is not a reduced ring.

Theorem 5.1.3. Let R be a commutative ring and $a \in R$ a nonzero very strongly irreducible element of R. Then $ann(a) \subseteq J(R)$. If $J(R) = 0$, then a is regular.

Proof. Let $x \in ann(a)$. Then $xa = 0$. Then for any $y \in R$ we have $y(xa) = 0$. So $a = a - yxa = a(1 - yx)$ implies $1 - yx \in U(R)$ since a is very strongly irreducible. Hence $x \in J(R)$. Thus $ann(a) \subseteq J(R)$. If $J(R) = 0$, then $ann(a) = 0$, and we conclude that a is regular. □

Corollary 5.1.4. Let R be a reduced commutative ring. Then a nonzero very strongly irreducible element f of $R[X]$ is regular. Also f cannot be written as the product of two polynomials each of positive degree, that is, f is regularly indecomposable.

Proof. By Theorem 3.1.3, $Nil(R[X]) = J(R[X])$. Thus R reduced gives $0 = Nil(R[X]) = J(R[X])$. Suppose $f = gh$, with $f, g \in R[X]$ with $deg(g), deg(h) \geq 1$. Then $f \cong g$ or $f \cong h$ since f is very strongly irreducible. Say $f \cong g$. Then $h \in U(R[X])$, with $h = c_0 + c_1X + \cdots + c_sX^s$, where $c_0 \in U(R)$ and each $c_i \in Nil(R)$ for $i = 1, \ldots, n$. But R is reduced which gives each $c_i = 0$ for $i = 1, \ldots, n$, so $deg(h) < 1$, a contradiction. It follows that f cannot be written as the product of two polynomials, each of positive degree. □

Corollary 5.1.5. If R is reduced and $R[X]$ is very strongly atomic, then R must be an integral domain. Thus R must be very strongly atomic.

Proof. Since $R[X]$ is very strongly atomic, by Corollary 5.1.4 every nonzero nonunit element of $R[X]$ is either very strongly irreducible or can be written as a product of very strongly irreducible elements that are regular. Thus R must be an integral domain. But clearly $R[X]$ atomic implies that R is atomic. □

In [19, Corollary 2.8] Coykendall and Trentham prove that if R is a reduced ring and $R[t]$ is strongly atomic, then R is strongly atomic. In this paper they took strongly atomic to be what we call very strongly atomic. Note that the above Corollary 5.1.5 sharpens their result by showing that their assumptions imply R is an integral domain.

Theorem 5.1.6. Let R be a commutative ring. Let $f \in R[X]$ be a zero divisor with $\deg f \geq 1$, then f is regularly decomposable.

Proof. Let $f \in R[X]$ be a zero divisor where $\deg(f) \geq 1$, then there exists a nonzero $g \in R[X]$ so that $fg = 0$. Then for any $h \in R[X]$, say $h = X^l$ for arbitrary $l \geq 1$, we have $h(fg) = 0$. Then

$$f = f + X^l(fg) = f(1 + X^l g)$$

gives a factorization of f with a factor of arbitrarily large degree. Thus f is decomposable since we assumed $deg(f) \geq 1$. □

Theorem 5.1.7. Let R be a commutative ring that is not reduced. Then every polynomial of degree at least 1 is regularly decomposable.

Proof. Suppose $f \in R[X]$ where $deg(f) \geq 1$, say $f = a_0 + a_1X + \cdots + a_nX^n$ where $a_n \neq 0$. Let $0 \neq \alpha \in R$ be nilpotent and $u = 1 + \alpha X + \cdots + \alpha X^{n+1}$. Then u is a unit in $R[X]$. Note that we can write

$$f = u(u^{-1}f)$$

so f is regularly decomposable if $\deg u^{-1}f \geq 1$. Suppose $u^{-1}f = c$ for some $c \in R$. Then we have

$$\begin{aligned} f &= (u^{-1}f)u \\ &= cu \\ &= c(1 + \alpha X + \cdots + \alpha X^n + \alpha X^{n+1}) \\ &= c + c\alpha X + \cdots + c\alpha X^n + c\alpha X^{n+1} \end{aligned}$$

where $0 \neq a_n = c\alpha$. But the term $c\alpha X^{n+1}$ must be zero, which implies $c\alpha = 0$, a contradiction. So $\deg u^{-1}f \geq 1$ and thus f is regularly decomposable. We can call such a decomposition the trivial decomposition of f. $\square$

Next we consider generalizations of an indecomposable polynomial to a polynomial ring with zero divisors using associate relations. We do this by using associate relations to give different types of indecomposable rings. Consider the following definitions.

Definition 5.1.8. Given a commutative ring R, $f \in R[X]$ is *indecomposable* (respectively *strongly indecomposable*, *very strongly indecomposable*) if $f = gh$ implies $g \sim_{R[X]} a$ or $h \sim_{R[X]} a$ (respectively $g \approx_{R[X]} a$ or $h \approx_{R[X]} a$, $g \cong_{R[X]} a$ or $h \cong_{R[X]} a$) for some $a \in R$.

We say f is *decomposable* (respectively *strongly decomposable*, *very strongly decomposable*) if f is not indecomposable (respectively strongly indecomposable, very strongly indecomposable). If a nonzero f is regularly indecomposable, then $f = gh$

implies $g \in R$ or $h \in R$ which implies $g \sim_{R[X]} a$ or $h \sim_{R[X]} a$ for some $a \in R$. Thus regularly indecomposable implies indecomposable. In fact since $g = a \cdot 1$ or $h = a \cdot 1$ for some $a \in R$ we actually have that regularly indecomposable implies strongly indecomposable. Note that very strongly indecomposable implies strongly indecomposable which implies indecomposable.

Suppose f is indecomposable with the property that whenever $f = gh$, then $g \sim_{R[X]} a$ or $h \sim_{R[X]} a$ with $a \in Reg(R)$. Then $(g) = (a)$ implies that $g = ak$ and $a = gl$ for some $k, l \in R[X]$. Then $a = gl = akl$ and since a is regular this implies $1 = kl$, so $k, l \in U(R[X])$. So f is very strongly indecomposable. Thus if f has the above property, then the notions of very strongly indecomposable, strongly decomposable, and indecomposable coincide. However, f is not regularly indecomposable. If a is not regular but $f = gh$ implies that $(g) = (a)$, then by reducing modulo a prime ideal we get that the coefficients of positive degree of g are nilpotent. Thus if R is a reduced ring, then $(g) = (a)$ implies that $g \in R$, and in this case f is regularly indecomposable. Below we give a diagram that shows the implications of the different types of indecomposable polynomials. Note that none of the arrows can be reversed.

$$\begin{array}{ccccc} & & \text{regularly indecomposable} & & \\ & & \Downarrow & & \\ \text{very strongly indecomposable} & \Longrightarrow & \text{strongly indecomposable} & \Longrightarrow & \text{indecomposable} \end{array}$$

Now if R is an integral domain, an element $a \in R$ is indecomposable as an element of $R[X]$ since $a = fg$ for $f, g \in R[X]$ implies both $\deg f = 0$ and $\deg g = 0$.

However, this is not true if R has zero divisors. Consider the following example of an element $a \in R$ that is not strongly indecomposable in $R[X]$.

Example 5.1.9. Let $R = \mathbb{Z}[B, C]/(5B, BC, 2C)$ where B, C are indeterminates over $\mathbb{Z}$. Denote the image of B and C by b, c respectively, so we can write $R = \mathbb{Z}[b, c]$. Note that $10 = (2 + bX)(5 + cX)$. Suppose $2 + bX \approx_{R[X]} a$ for some $a \in R$. Then $2 + bX = af$ for some $f \in U(R[X])$ with $f = a_0 + a_1X + \cdots + a_nX^n$. So a_0 is a unit and $a_1, \ldots, a_n$ are nilpotent. Then $b = aa_1$ where a_1 is nilpotent, thus b is nilpotent. Then there exists an $n \in \mathbb{N}$ such that $b^n = \overline{0}$. If we consider the prime ideal $(5, C) \supseteq (5B, BC, 2C)$ then $B \notin (5, C)$ so we cannot have $B^n \in (5B, BC, 2C)$. So $2 + bX \not\approx_{R[X]} a$ for some $a \in R$. If $5 + cX \approx a$ for some $a \in R$, then c is nilpotent, another contradiction. So neither $2 + bX$ or $5 + cX$ is strongly associated to any $a \in R$. It follows that 10 is strongly decomposable in $R[X]$.

So we see that constant polynomials are not necessarily indecomposable. We showed in example 5.15 that 10 in $R = \mathbb{Z}[B, C]/(5B, BC, 2C)$ was not strongly indecomposable, in a similar fashion we can show that it is not indecomposable, thus it is not indecomposable of any type. In [12] Anderson and Valdes-Leon give an example of an element that is very strongly irreducible in a ring R but not even strongly irreducible in $R[X]$. We use this same element to give a constant polynomial that is indecomposable but not strongly indecomposable.

Example 5.1.10. [12, Example 6.1] A constant polynomial that is indecomposable but not strongly indecomposable.

Let $R = \mathbb{Z}_{(2)}(+)\mathbb{Z}_4$ be the idealization where $\mathbb{Z}_{(2)}$ denotes the localization of $\mathbb{Z}$ at the prime ideal (2). Let $a = (0, \bar{1})$ and $f = (1, \bar{0}) + (2, \bar{0})X$. Then $af = (0, \bar{1})\big((1, \bar{0}) + (2, \bar{0})X\big) = (0, \bar{1}) + (0, \bar{2})X$ and $af^2 = (0, \bar{1})\big((1, \bar{0}) + (4, \bar{0})X + (4, \bar{0})X^2\big) = (0, \bar{1})$. So $a = af^2 = (af)f$. Note that $af \sim_{R[X]} a$ so a is indecomposable.

Suppose $a \approx_{R[X]} af$. Then $af = au$ for some $u \in U(R[X])$, say $u = r_0 + r_1X + \cdots + r_sX^s$. Note that the $r_i \in Nil(R) = 0(+)\mathbb{Z}_4$, so $ar_i = 0$ for all $1 \leq i \leq s$. Then we have $af = au = ar_0 \in R$, a contradiction. So $a \not\approx af$ implies that a is not strongly indecomposable.

Example 5.1.11. A constant polynomial that is strongly indecomposable but not very strongly indecomposable.

Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ where $\mathbb{Z}_2$ is the integers modulo 2. Consider $e = (\bar{1}, \bar{0}) \in R$. Since e is strongly irreducible in R and $R[X]$, $e = fg$ implies that $f = ue$ or $g = ue$ where $u \in U(R[X]) = \{1\}$. So e is strongly indecomposable. However, $e = e \cdot e$ but $e \not\cong_{R[X]} a \in R$, so it is not very strongly indecomposable.

5.2 When is an Indeterminate Indecomposable?

In Theorem 4.1.2 we saw that X is irreducible in $R[X]$ if and only if the ring R is indecomposable. Now we can consider when X is indecomposable in $R[X]$. The following theorem tells us that X is indecomposable if and only if X is irreducible.

Theorem 5.2.1. Let R be a commutative ring. Then X is indecomposable in $R[X]$ if and only if X is irreducible.

Proof. ($\Longrightarrow$) Suppose X is indecomposable and let $X = fg$ with $f = a_0 + a_1X + \cdots +$

a_nX^n and $g = b_0 + b_1X + \cdots + b_mX^m$. Then $f \sim_{R[X]} a$ or $g \sim_{R[X]} a$ for some $a \in R$. Suppose $f \sim_{R[X]} a$, then $(f) = (a)$ and $f = ah$ and $a = fk$ for some $h, k \in R[X]$. Then $X = ahg$. So $R = c(X) = c(ahg) = ac(hg)$ gives that $a \in U(R)$, and hence $f \in U(R[X])$. So X is irreducible, in fact it is very strongly irreducible.

($\Longleftarrow$) Let $X = fg$ and suppose $X \sim f$. Then $f = Xh$ for some $h \in R[X]$ and $X = fg = Xhg$ implies $1 = hg$ so $g \in U(R[X])$. Then $g \sim_{R[X]} 1$. Similarly, if $X \sim g$ then $f \sim_{R[X]} 1$, so X is indecomposable. $\square$

Similarly, we can show that X is irreducible if and only if it is strongly indecomposable and also very strongly indecomposable. This gives some evidence that the weakest generalization of indecomposable is appropriate. If instead we take X to be regularly indecomposable then we also have that X is irreducible, but we need the additional condition that R is a reduced ring.

Theorem 5.2.2. Let R be a reduced ring. Then X is regularly indecomposable if and only if X is irreducible.

Proof. ($\Longrightarrow$) Let X be reguarly indecomposable and let $X = fg$ with $f = a_0 + a_1X + \cdots + a_nX^n$ and $g = b_0 + b_1X + \cdots + b_mX^m$. Then since f is regularly indecomposable, one of f or g has degree less than 1, say f. Then $f = a_0$. So the coefficient of X in fg is given by $1 = a_0b_1$. So $a_0 \in U(R)$, so f is a unit, thus X is irreducible.

($\Longleftarrow$) Suppose X is irreducible, and let $X = fg$. Suppose $X \sim f$ then $f = Xh$ for some $h \in R[X]$ and $X = fg = Xhg$ implies $1 = hg$, so $g \in U(R[X])$, but R is reduced, so $g \in U(R)$ and $\deg g < 1$, Similarly, if $X \sim g$, then $\deg f < 1$. $\square$

Now we would like for X to be indecompsable if and only if the ring R is indecomposable. If we combine the results from Theorem 5.1.7 and Theorem 4.1.2 we obtain the following result.

Theorem 5.2.3. For a commutative ring R, the following are equivalent:

1. X is irreducible in $R[X]$,

2. X is indecomposable in $R[X]$, and

3. R is indecomposable.

Proof. The result follows since $1 \iff 2$ by Theorem 5.1.7 and $1 \iff 3$ by Theorem 4.1.2. $\square$

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