Some Remarks on Uniqueness

Polynomial Rings and My Journey in Pure Mathematics

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OUTLINE

I. Introduction

II. About Me

III. Graduate School

IV. Research

V. Jobs

VI. Conclusion

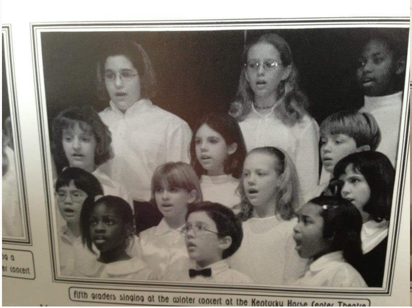
Introduction

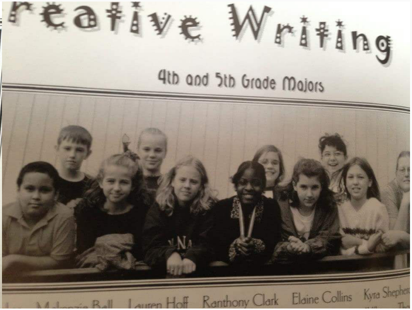
In mathematics, the term ‘unique’ is used to indicate that there is exactly one object that exists with a certain property.

About Me

School for the Creative and Performing Arts

- Attended from 4th to 8th grade
- Creative Writing major
- Band minor (clarinet)
- Took electives in art, music, dance, drama, etc





PAUL LAURENCE DUNBAR

Math, Science, and Technology Center (MSTC)

- 55/2500 incoming high school freshmen selected each year
- Dismissed after year 1!
- Discouraged me from pursuing a STEM career

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Writing

- Worked on school's literary magazine
- Went to a summer journalism camp at Western Kentucky University
- Worked for school newspaper *The Lamplighter* as photo editor

UNIVERSITY OF KENTUCKY

Freshman Year

- Started as Biology Pre-Med Major

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- Changed my major to English Pre-Med
- Began leading math study groups twice a week on campus

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Junior Year

- Added an African-American Studies minor
- Changed my major to dual degree English and Mathematics

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Junior Year

- Added an African-American Studies minor
- Changed my major to dual degree English and Mathematics

Super Senior Year

- Applied to graduate school for a Master's in Math Education

Graduate School

EASTERN KENTUCKY UNIVERSITY

The Plan

- Enrolled in Master's of Science in Mathematical Sciences Program (2 years)
- Planned on Applying to Math for America and then PhD Programs in Math Education

How Did it Get Paid For?

- Full tuition plus a living stipend as a Graduate Assistant

Graduate Teaching Assistant

- Assisted professors in discussion sections, worked in math tutoring lab, led my own lecture

Kicking off his last office hours before the first major paper is due, the grad student regrets not setting up an online appointment system.

UNIVERSITY OF IOWA

Department of Mathematics

- PhD Programs in Mathematics and Applied Mathematics
- Average Time Length: 5-6 years
- Every PhD student accepted is fully funded as a Graduate Teaching Assistant (average stipend of $18,264 to $19,133 for 2015-2016 school year)
- Graduate College and National Fellowships are also available
- Common Job Outlooks: Academia and Industry

Peeking into a faculty panel speaking to the admits, the grad student ruefully chuckles at their wildly cheerful overview of the program.

Application Process Required Materials

- Unofficial transcripts
- GRE General Test scores
- Coursework information
- Statement of purpose
- 3 Letters of recommendation
- Resume/CV (optional)
- Deadline: January 15, 2018

Prospective Student Weekend

- Visit campus and get to know faculty and students in department
- All expenses paid
- Ask a lot of questions!
- Decision deadline: April 15, 2018

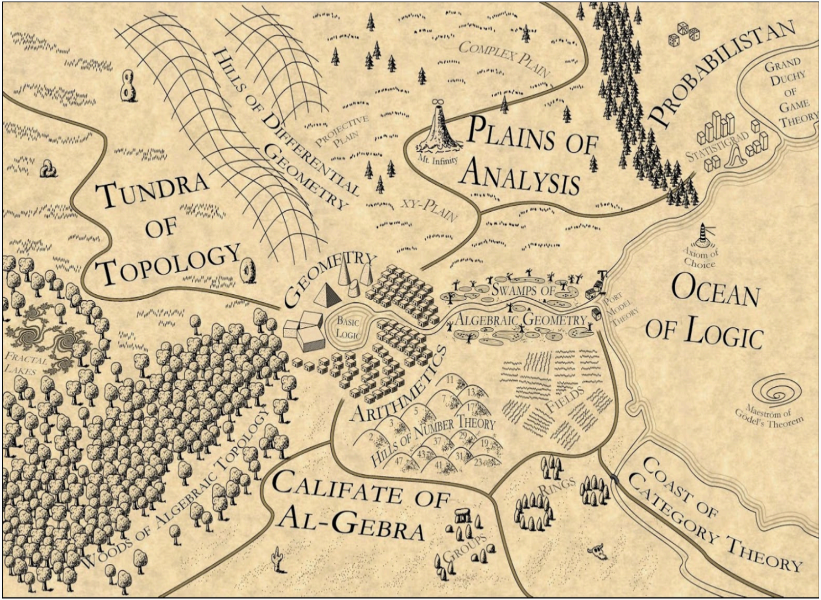
Wandering by eager admits on their prospective visit, the grad student sees a new crop of souls unaware that they, too, will be shattered.

What do you do for 5-6 years?

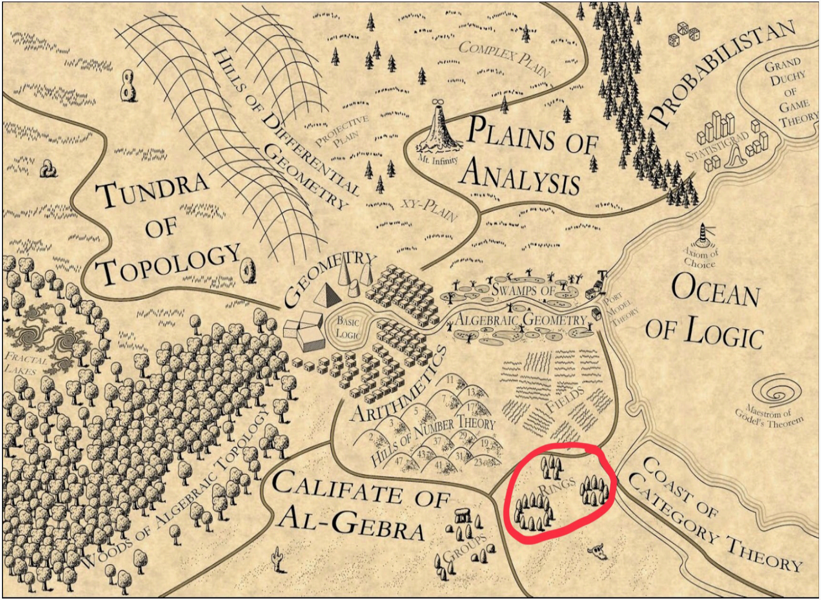
- **Year 1:** Complete first year sequences
- Take Qualifying Exams the Summer before Year 2 over first year coursework
- **Year 2:** Complete second year sequences (look for an advisor)
- **Year 3:** Upper level coursework
- Pass Qualifying Exam by Winter of Year 3
- **Year 4:** Complete Coursework
- Pass Comprehensive Exam by Winter of Year 4
- **Years 5-6:** Work on Writing Dissertation (apply for jobs)

Research

RESEARCH IN PURE MATHEMATICS



RESEARCH IN PURE MATHEMATICS



Definition

A **ring** R is a set equipped with two binary operations $+$ and $\cdot$ that satisfies three sets of axioms.

<u>Axiom 1</u>: **R is an abelian group under addition.**

- **associative:** $(a+b)+c = a+(b+c)$ for all $a, b, c \in R$
- **commutative:** $a+b = b+a$ for all $a, b \in R$
- **additive identity:** There is an element $0 \in R$ so that $a+0 = a$ for all $a \in R$
- **additive inverse:** For each $a \in R$ there exists an $-a \in R$ so that $a+(-a) = 0$

Axiom 2: R **is a monoid under multiplication**

- **associative:** $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$
- **multiplicative identity:** There is an element 1 in R such that $a \cdot 1 = a$ and $1 \cdot a = a$ for all $a \in R$

Axiom 3: **Multiplication in** R **distributes over addition**

- **left distributivity:** $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in R$
- **right distributivity:** $(b + c) \cdot a = b \cdot a + c \cdot a$ for all $a, b, c \in R$

Example 1

- The integers $\mathbb{Z}$ are a ring under the usual addition and multiplication
- $R[X]$ is a ring under polynomial addition and multiplication

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- $R[X]$ is a ring under polynomial addition and multiplication
  - ex. Let $f(x) = x + 3$ and $g(x) = 2x + 5 \in \mathbb{R}[X]$ then

$$f(x) + g(x) = 3x + 8 \text{ and } f(x) \cdot g(x) = 2x^2 + 11x + 15$$

- The set of even integers $2\mathbb{Z}$ is not a ring, why?

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- The integers $\mathbb{Z}$ are a ring under the usual addition and multiplication
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- The set of even integers $2\mathbb{Z}$ is not a ring, why?
  - There is no multiplicative identity element!

Properties of Rings

Definition

R is a **commutative** ring if the multiplication in R is commutative, that is if $a \cdot b = b \cdot a$ for all $a, b \in R$

Example 2

- The integers $\mathbb{Z}$ are a commutative ring, for example $2 \cdot 3 = 3 \cdot 2$
- The set of 2×2 matrices with real number coefficients is not a commutative ring

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 3 & 10 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ 3 & 3 \end{pmatrix}$$

Example 3

- The integers modulo n $\mathbb{Z}/n\mathbb{Z} = \{[0], [1], [2], \ldots, [n-1]\}$ are a commutative ring
- In $\mathbb{Z}/4\mathbb{Z} = \{[0], [1], [2], [3]\}$ every integer is identified with its congruence class based on its remainder when you divide by 4. So $7 = [3]$ and $16 = [0]$. Which congruence class does 25 belong to?

Properties of Rings

Definition

R is an **integral domain** if $a \cdot b = 0$ implies that $a = 0$ or $b = 0$.

Example 3

- Solve $2x = 0$ for $x \in \mathbb{Z}$

Properties of Rings

Definition

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Example 3

- Solve $2x = 0$ for $x \in \mathbb{Z}$
- Solve $2x^3 + 5x^2 + 6x = 0$ with $2x^3 + 5x^2 + 6x \in \mathbb{R}[X]$

$$\begin{aligned} 0 &= x^3 + 5x^2 + 6x \\ &= x(x^2 + 5x + 6) \\ &= x(x+2)(x+3) \end{aligned}$$

Then $x = 0$ or $x + 2 = 0$ or $x + 3 = 0$ so $x = 0, -2,$ or -3.

Properties of Rings

Definition

An element $a \in R$, is called a **zero divisor** if there exists a nonzero element b so that $a \cdot b = 0$.

Example

- Solve $2x = 0$ and
- Solve $2x = 2y$ for $x, y \in \mathbb{Z}/4\mathbb{Z}$

$\cdot$	[0]	[1]	[2]	[3]
[0]	[0]	[0]	**[0]**	[0]
[1]	[0]	[1]	[2]	[3]
[2]	**[0]**	[2]	**[0]**	[2]
[3]	[0]	[3]	[2]	[1]

Properties of Rings

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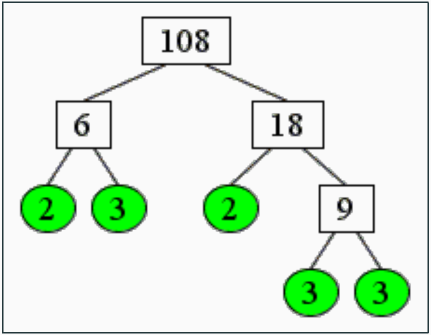
[1]	[0]	[1]	[2]	[3]
[2]	**[0]**	[2]	**[0]**	[2]
[3]	[0]	[3]	[2]	[1]

- 2 is a zero divisor in $\mathbb{Z}/4\mathbb{Z}$
- $2 \cdot 0 = 2 \cdot 0$; $2 \cdot 1 = 2 \cdot 1$; $2 \cdot 1 = 2 \cdot 3$ but $1 \neq 3$! (no cancellation)

What is Factorization Theory?



$$a^2 + 2ab + b^2 = (a+b)^2$$

$$a^2 - 2ab + b^2 = (a-b)^2$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

UNIQUE FACTORIZATION

Definition

An element a of a ring R is a **unit** if there exists a $b \in R$ so that $ab = 1$.

Example

- 2 is a unit in $\mathbb{R}$
- 2 is not a unit in $\mathbb{Z}$

Definition

A **unique factorization domain** (UFD) is an integral domain R where every non-zero non-unit element can be written as a product of irreducible elements uniquely up to order and units

Example

- The integers $\mathbb{Z}$ are a UFD; ex. $70 = 2 \cdot 5 \cdot 7$

UNIQUE FACTORIZATION

Unique Factorization is the most ideal factorization property a ring can have!

In a UFD...

- If $x = p_1 \cdot p_2 \cdots p_n$ is a factorization of x into irreducibles and $x = q_1 \cdot q_2 \cdots q_m$ is another factorization of x into irreducibles then $n = m$ and after a reordering the p_i and q_j only differ by a unit
- There is only one way to factor any nonzero non-unit element

FACTORIZATION IN INTEGRAL DOMAINS

Definition

An integral domain R is **atomic** if each nonzero non-unit can be written as the finite product of atoms, i.e. irreducible elements.

Definition

An integral domain satisfies $ACCP$, the **ascending chain condition on principal ideals**, if there does not exists an infinitely strictly ascending chain of principal ideals in R.

Definition

An integral domain is a **bounded factorization domain** (BFR) if R is atomic and there exists a bound on the length of factorizations into products of irreducibles for every nonzero non-unit of R

Example

- $R[X^2, X^3]$ is a bounded factorization domain

Definition

An integral domain R is a **half-factorial domain** (HFD) if R is atomic and each factorization of a nonzero non-unit of R into a product of irreducibles has the same length

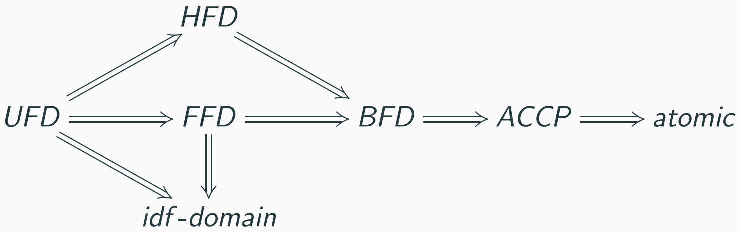
Example

- In $R[X^2, X^3]$, $X^6 = X^2 \cdot X^2 \cdot X^2 = X^3 \cdot X^3$ is not a HFD

Definition

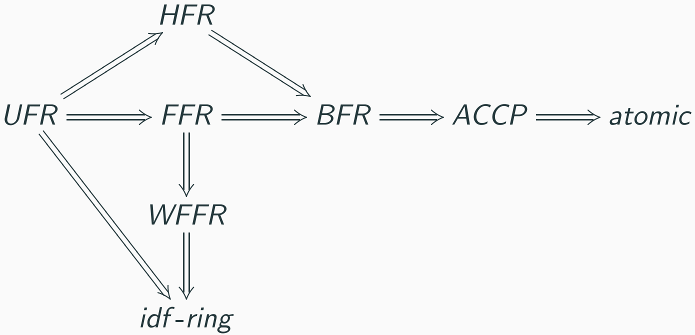
An integral domain is a finite factorization domain (FFD) if every nonzero non-unit has a finite number of factorizations up to order and associates.

FACTORIZATION IN INTEGRAL DOMAINS



FACTORIZATION IN COMMUTATIVE RINGS WITH ZERO DIVISORS

The theory from integral domains generalizes to commutative rings with zero divisors.

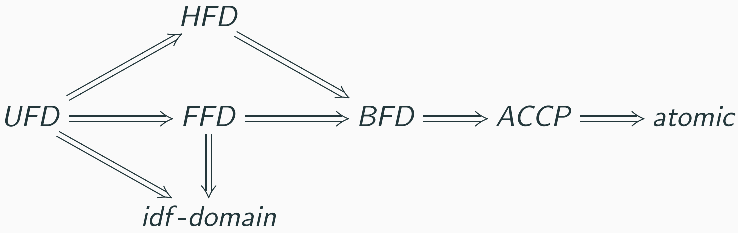


MY RESEARCH

Thesis Title

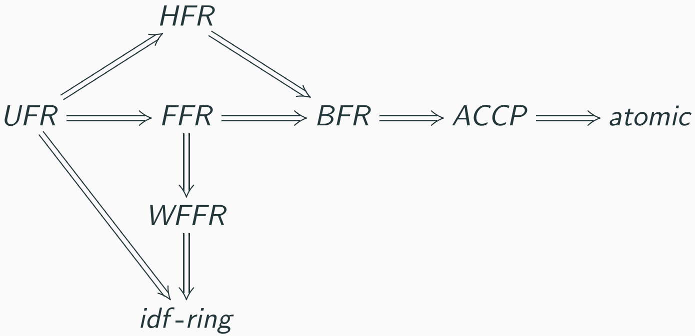
"Factorization in Polynomial Rings with Zero Divisors"

FACTORIZATION IN POLYNOMIAL RINGS



Property	UFD	HFD	FFD	idf-domain	BFD	ACCP	atomic
R	yes	yes	yes	yes	yes	yes	yes
R[X]	yes	no	yes	no	yes	no	no

FACTORIZATION IN POLYNOMIAL RINGS



Property	UFR	HFR	FFR	WFFR	idf	BFR	ACCP	atomic
R	yes	yes	yes	yes	yes	yes	yes	yes
R[X]	no	no	no	no	no	no	no	no

FACTORIZATION IN POLYNOMIAL RINGS

Theorem

For a commutative ring R the following are equivalent:

1. X is irreducible in $R[X]$
2. X is indecomposable in $R[X]$
3. R is indecomposable

FACTORIZATION IN POLYNOMIAL RINGS

Theorem

Let R be a commutative ring. Then X is a product of n atoms if and only if R is a direct product of n indecomposable rings.

Corollary

When X is a finite product of atoms, the factorization is unique up to order and associates.

Question

When does X^n have unique factorization?

Example

X^n does not have unique factorization in $\mathbb{Z}_4[X]$,

- $X^2 = X \cdot X = (X+2)(X+2)$
- $X^3 = X \cdot X \cdot X = X(X+2)(X+2)$
- $X^4 = X \cdot X \cdot X \cdot X = (X^2+2)(X^2+2)$
- $X^5 = X \cdot X \cdot X \cdot X \cdot X = X(X^2+2)(X^2+2)$

FACTORING POWERS OF INDETERMINATES

Let $L(X^n)$ and $l(X^n)$ represent the longest and shortest lengths of a factorization of X^n into atoms in $\mathbb{Z}_4[X]$ and $\rho(X^n) = L(X^n)/l(X^n)$

Theorem

In $\mathbb{Z}_4[x]$, $L(X^n) = l(X^n)$ if $n = 1$ and for $n > 1$ $L(X^n) = n$,

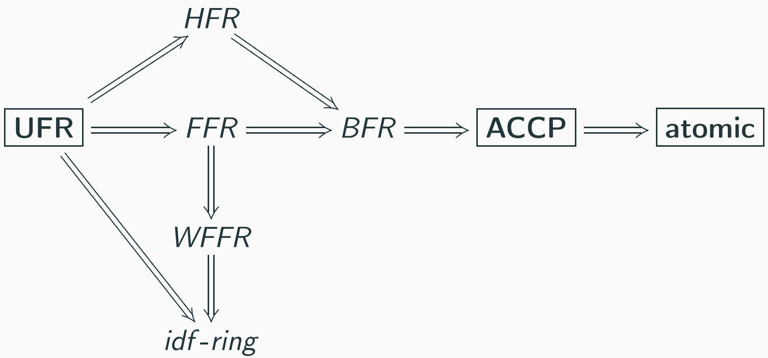
$$l(X^n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases} \text{ and } \rho(X^n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ n/3 & \text{if } n \text{ is odd} \end{cases}.$$

How do we prove this?

Some Types of Unique Factorization Rings

- Bouvier-Galovich Unique Factorization Ring
- Fletcher Unique Factorization Ring

PROGRESS



IMPLICATIONS

Filling out the “Significance” section of a grant proposal, the grad student gets a chance to exercise his creative writing skills.

Jobs

JOBS

Academia? Industry? Something Else Entirely?

Entering the house owned by a friend working in the private sector, the grad student anxiously reassesses many of his life choices.

JOBS

Sources

- www.mathjobs.org
- eims.ams.org/jobs
- higheredjobs.com
- Society for Industrial and Applied Mathematics (SIAM) www.siam.org

Industry Examples

- Financial and Insurance Sector
- Security (Department of Defense (DoD), National Security Agency (NSA))
- Data Analysis, Modeling

JOBS

Academic Examples

- Tenure-track professor (Research University)
  - ex. University of Iowa
  - ex. Iowa State University
- Tenure-track professor (Liberal Arts College or Mid-Tier Research University)
  - ex. William Penn University
  - ex. Grinnell College
  - ex. Drake University
  - ex. University of Northern Iowa
- Community College Professor
- Temporary Positions (Post-docs, Visiting Professors, Adjuncts, Lecturers)

LONG TERM

Academic

- Spring 2019: Graduation from University of Iowa
- Post-doc (2-3 years)
- Professor at Liberal Arts College or Mid-Tier Research University
- Long Term Goal: Administration

Community

- Non-Profit Work
- Curriculum Development for Vocational Mathematics and/or College-Readiness

Conclusion

Things to Take Away

- There is no ‘right’ way to achieve your goals
- Consider graduate school in mathematics
- Pure and Applied Math are two sides to the same coin
- Research can be very exciting
- There are many career options with a degree in mathematics
- Intangible skills are invaluable
- Mentors and a support team are important
- Understand what makes you unique

Questions?

CONTACT

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Slides

www.RanthonyEdmonds.com /conferences-and-presentations.html